Acyclic Edge Coloring of 2-degenerate Graphs

Manu Basavaraju[*]   L. Sunil Chandran[†]

Abstract

An *acyclic* edge coloring of a graph is a proper edge coloring such that there are no bichromatic cycles. The *acyclic chromatic index* of a graph is the minimum number k such that there is an acyclic edge coloring using k colors and is denoted by $a'(G)$. A graph is called 2-*degenerate* if any of its induced subgraph has a vertex of degree at most 2. The class of 2-*degenerate graphs* properly contain *series-parallel graphs*, *outerplanar graphs*, *non-regular subcubic graphs*, *planar graphs of girth at least 6* and *circle graphs of girth at least 5* as subclasses. It was conjectured by Alon, Sudakov and Zaks (and much earlier by Fiamcik) that $a'(G) \leq \Delta + 2$, where $\Delta = \Delta(G)$ denotes the maximum degree of the graph. We prove the conjecture for 2-*degenerate* graphs. In fact we prove a stronger bound: we prove that if G is a 2-degenerate graph with maximum degree Δ, then $a'(G) \leq \Delta + 1$.

Keywords: Acyclic edge coloring, acyclic edge chromatic number, 2-degenerate graphs, series-parallel graphs, outer planar graphs.

1   Introduction

All graphs considered in this paper are finite and simple. A proper *edge coloring* of $G = (V, E)$ is a map $c : E \to C$ (where C is the set of available *colors*) with $c(e) \neq c(f)$ for any adjacent edges e, f. The minimum number of colors needed to properly color the edges of G, is called the chromatic index of G and is denoted by $\chi'(G)$. A proper edge coloring c is called acyclic if there are no bichromatic cycles in the graph. In other words an edge coloring is acyclic if the union of any two color classes induces a set of paths (i.e., linear forest) in G. The *acyclic edge chromatic number* (also called *acyclic chromatic index*), denoted by $a'(G)$, is the minimum number of colors required to acyclically edge color G. The concept of *acyclic coloring* of a graph was introduced by Grünbaum [15]. The *acyclic chromatic index* and its vertex analogue can be used to bound other parameters like *oriented chromatic number* and *star chromatic number* of a graph, both of which have many practical applications, for example, in wavelength routing in optical networks ([5], [16]). Let $\Delta = \Delta(G)$ denote the maximum degree of a vertex in graph G. By Vizing's theorem, we have $\Delta \leq \chi'(G) \leq \Delta + 1$(see [11] for proof). Since any acyclic edge coloring is also proper, we have $a'(G) \geq \chi'(G) \geq \Delta$.

It has been conjectured by Alon, Sudakov and Zaks [3] (and much earlier by Fiamcik [12]) that $a'(G) \leq \Delta + 2$ for any G. Using probabilistic arguments Alon, McDiarmid and Reed [2] proved that $a'(G) \leq 60\Delta$. The best known result up to now for arbitrary graph, is by Molloy and Reed [17] who showed that $a'(G) \leq 16\Delta$. Muthu, Narayanan and Subramanian [18] proved that $a'(G) \leq 4.52\Delta$ for graphs G of girth at least 220 (*Girth* is the length of a shortest cycle in a graph).

Though the best known upper bound for general case is far from the conjectured $\Delta + 2$, the conjecture has been shown to be true for some special classes of graphs. Alon, Sudakov and Zaks [3] proved that there exists a constant k such that $a'(G) \leq \Delta + 2$ for any graph G whose girth is at least $k\Delta \log \Delta$. They also proved that $a'(G) \leq \Delta + 2$ for almost all Δ-regular graphs. This result was improved by Nešetřil and Wormald [22] who showed that for a random Δ-regular graph $a'(G) \leq \Delta + 1$. Muthu, Narayanan and Subramanian proved the conjecture for grid-like graphs [19]. In fact they gave a better bound of $\Delta + 1$ for these class of graphs. From Burnstein's [9] result it follows that the conjecture is true for subcubic graphs. Skulrattankulchai [23] gave a polynomial time algorithm to color a subcubic graph using $\Delta + 2 = 5$ colors.

Determining $a'(G)$ is a hard problem both from theoretical and algorithmic points of view. Even for the simple and highly structured class of complete graphs, the value of $a'(G)$ is still not determined exactly. It has also been shown by

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[*]Computer Science and Automation department, Indian Institute of Science, Bangalore- 560012, India. manu@csa.iisc.ernet.in

[†]Computer Science and Automation department, Indian Institute of Science, Bangalore- 560012, India. sunil@csa.iisc.ernet.in

Alon and Zaks [4] that determining whether $a'(G) \leq 3$ is NP-complete for an arbitrary graph G. The vertex version of this problem has also been extensively studied (see [15], [9], [8]). A generalization of the acyclic edge chromatic number has been studied: The *r-acyclic edge chromatic number* $a'_r(G)$ is the minimum number of colors required to color the edges of the graph G such that every cycle C of G has at least $\min\{|C|, r\}$ colors (see [13], [14]).

Our Result: We prove the conjecture for 2-degenerate graphs. A graph G is called k-*degenerate* if any induced subgraph of G, has a vertex of degree at most k. For example, planar graphs are 5-degenerate, forests are 1-degenerate. The earliest result on acyclic edge coloring of 2-degenerate graphs was by Card and Roditty [10], where they proved that $a'(G) \leq \Delta + k - 1$, where k is the maximum edge connectivity, defined as $k = \max_{u,v \in V(G)} \lambda(u, v)$, where $\lambda(u, v)$ is the edge- connectivity of the pair u,v. Note that here k can be as high as Δ. Muthu,Narayanan and Subramanian [20] proved that $a'(G) \leq \Delta + 1$ for outerplanar graphs which are a subclass of 2-degenerate graphs and posed the problem of proving the conjecture for 2-degenerate graphs as an open problem. In fact they have informed us that very recently they have also derived an upper bound of $\Delta + 1$ for series-parallel graphs [21], which is a slightly bigger subclass of 2-degenerate graphs. Connected non-regular subcubic graphs are 2-degenerate graphs with $\Delta = 3$. Recently Basavaraju and Chandran [6] proved that connected non-regular subcubic graph can be acyclically edge colored using $\Delta + 1 = 4$ colors. Another two interesting subclasses of 2-degenerate graphs are *planar graphs of girth 6* and *circle graphs of girth 5* [1]. As far as we know, nothing much is known about the acyclic edge chromatic number of these graphs. In this paper, we prove the following theorem,

Theorem 1. *Let G be a 2-degenerate graph with maximum degree Δ, then $a'(G) \leq \Delta + 1$.*

Our result is tight since there are 2-degenrate graphs which require $\Delta + 1$ colors (e.g., cycle, non-regular subcubic graphs, etc.). Most of the work in this field has been nonconstructive, using probabilistic methods. In contrast, our proof is constructive and yields an efficient polynomial time algorithm. It is easy to see that its complexity is $O(\Delta n^2)$. (We have presented the proof in a non-algorithmic way. But it is easy to extract the underlying algorithm from it.)
Remark: It may be noted that though $a'(G) \leq \Delta + 1$ for 2-degenerate graphs, it is not so in general. In fact every Δ-regular graph on $2n$ vertices with $\Delta > n$ requires at least $\Delta + 2$ colors to be acyclically edge colored (See [7]).

2   Preliminaries

Let $G = (V, E)$ be a simple, finite and connected 2-degenerate graph of n vertices and m edges. Let $x \in V$. Then $N_G(x)$ will denote the neighbours of x in G. For an edge $e \in E$, $G - e$ will denote the graph obtained by deletion of the edge e. For $x, y \in V$, when $e = (x, y) = xy$, we may use $G - \{xy\}$ instead of $G - e$. Let $c : E \rightarrow \{1, 2, \ldots, k\}$ be an *acyclic edge coloring* of G. For an edge $e \in E$, $c(e)$ will denote the color given to e with respect to the coloring c. For $x, y \in V$, when $e = (x, y) = xy$ we may use $c(x, y)$ instead of $c(e)$. For $S \subseteq V$, we denote the induced subgraph on S by $G[S]$.

Partial Coloring: Let H be a subgraph of G. Then an acyclic edge coloring c' of H is also a partial coloring of G. Note that H can be G itself. Thus a coloring c of G itself can be considered a partial coloring. A coloring c of G is said to be a proper partial coloring if c is proper. A proper partial coloring c is called acyclic if there are no bichromatic cycles in the graph. Sometimes we also use the word valid coloring instead of acyclic coloring. Note that with respect to a partial coloring c, $c(e)$ may not be defined for an edge e. So, whenever we use $c(e)$, we are considering an edge e for which $c(e)$ is defined, though we may not always explicitly mention it.

Let c be a partial coloring of G. We denote the set of colors in the partial coloring c by $C = \{1, 2, \ldots, \Delta + 1\}$. For any vertex $u \in V(G)$, we define $F_u(c) = \{c(u, z) | z \in N_G(u)\}$. For an edge $ab \in E$, we define $S_{ab}(c) = F_b - \{c(a, b)\}$. Note that $S_{ab}(c)$ need not be the same as $S_{ba}(c)$. We will abbreviate the notation to F_u and S_{ab} when the coloring c is understood from the context.

To prove the main result, we plan to use contradiciton. Let G be the minimum counter example for the statement in $Theorem$ 1. Let $G = (V, E)$ be a graph on m edges where $m \geq 1$. We will remove an edge e from G and get a graph $G' = (V, E')$. By the minimality of G, the graph G' will have an acyclic edge coloring $c : E' \rightarrow \{1, 2, \ldots, \Delta + 1\}$. Our intention will be to extend the coloring c of G' to G by assigning an appropriate color for the edge e thereby contrdicting the assumption that G is a minimum couter example.

The following defintions arise out of our attempt to understand what may prevent us from extending a partial coloring of $G - e$ to G.

Maximal bichromatic Path: An (α,β)-maximal bichromatic path with respect to a partial coloring c of G is a maximal path consisting of edges that are colored using the colors α and β alternatingly. An (α,β,a,b)-maximal bichromatic path is an (α,β)-maximal bichromatic path which starts at the vertex a with an edge colored α and ends at b. We emphasize that the edge of the (α,β,a,b)-maximal bichromatic path incident on vertex a is colored α and the edge incident on vertex b can be colored either α or β. Thus the notations (α,β,a,b) and (α,β,b,a) have different meanings. Also note that any maximal bichromatic path will have at least two edges. The following fact is obvious from the definition of proper edge coloring:

Fact 1. *Given a pair of colors α and β of a proper coloring c of G, there can be at most one maximal (α,β)-bichromatic path containing a particular vertex v, with respect to c.*

A color $\alpha \neq c(e)$ is a *candidate* for an edge e in G with respect to a partial coloring c of G if none of the adjacent edges of e are colored α. A candidate color α is *valid* for an edge e if assigning the color α to e does not result in any bichromatic cycle in G.

Let $e = (a,b)$ be an edge in G. Note that any color $\beta \notin F_a \cup F_b$ is a candidate color for the edge ab in G with respect to the partial coloring c of G. But β may not be valid. What may be the reason? It is clear that color β is not *valid* if and only if there exists $\alpha \neq \beta$ such that a (α,β)-bichromatic cycle gets formed if we assign color β to the edge e. In other words, if and only if, with respect to coloring c of G there existed a (α,β,a,b) maximal bichromatic path with α being the color given to the first and last edge of this path. Such paths play an important role in our proof. We call them *critical paths*. It is formally defined below:

Critical Path: Let $ab \in E$ and c be a partial coloring of G. Then a (α,β,a,b) maximal bichromatic path which starts out from the vertex a via an edge colored α and ends at the vertex b via an edge colored α is called an (α, β, ab) critical path. Note that any critical path will be of odd length. Moreover the smallest length possible is three.

An obvious strategy to extend a valid partial coloring c of G would be to try to assign one of the candidate colors to an uncolored edge e. The condition that a candidate color being not valid for the edge e is captured in the following fact.

Fact 2. *Let c be a partial coloring of G. A candidate color β is not valid for the edge $e = (a,b)$ if and only if $\exists \alpha \in S_{ab} \cap S_{ba}$ such that there is a (α, β, ab) critical path in G with respect to the coloring c.*

If all the candidate colors turn out to be *invalid*, we try to *slightly modify* the partial coloring c in such a way that with respect to the modified coloring, one of the candidate colors becomes valid. An obvious way to modify is to recolor an edge so that some critical paths are *broken* and a candidate color becomes valid. Sometimes we resort to a slightly more sophisticated strategy to modify the coloring namely *color exchange* defined below:

Color Exchange: Let c be a partial coloring of G. Let $u, i, j \in V(G)$ and $ui, uj \in E(G)$. We define *Color Exchange* with respect to the edge ui and uj, as the modification of the current partial coloring c by exchanging the colors of the edges ui and uj to get a partial coloring c', i.e., $c'(u,i) = c(u,j)$, $c'(u,j) = c(u,i)$ and $c'(e) = c(e)$ for all other edges e in G. The color exchange with respect to the edges ui and uj is said to be proper if the coloring obtained after the exchange is proper. The color exchange with respect to the edges ui and uj is *valid* if and only if the coloring obtained after the exchange is acyclic. The following fact is obvious:

Fact 3. *Let c' be the partial coloring obtained from a valid partial coloring c by the color exchange with respect to the edges ui and uj. Then the partial coloring c' will be proper if and only if $c(u,i) \notin S_{uj}$ and $c(u,j) \notin S_{ui}$.*

The color exchange is useful in breaking some critical paths as is clear from the following lemma:

Lemma 1. *Let $u, i, j, a, b \in V(G)$, $ui, uj, ab \in E$. Also let $\{\lambda, \xi\} \in C$ such that $\{\lambda, \xi\} \cap \{c(u,i), c(u,j)\} \neq \emptyset$ and $\{i, j\} \cap \{a, b\} = \emptyset$. Suppose there exists an (λ,ξ,ab)-critical path that contains vertex u, with respect to a valid partial coloring c of G. Let c' be the partial coloring obtained from c by the color exchange with respect to the edges ui and uj. If c' is proper, then there will not be any (λ,ξ,ab)-critical path in G with respect to the partial coloring c'.*

Proof. Firstly, $\{\lambda, \xi\} \neq \{c(u,i), c(u,j)\}$. This is because, if there is a (λ,ξ,ab)-critical path that contains vertex u, with respect to a valid partial coloring c of G, then it has to contain the edge ui and uj. Since $i \notin \{a, b\}$, vertex i is an internal

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vertex of the critical path which implies that both the colors λ and ξ (that is $c(u, i)$ and $c(u, j)$) are present at vertex i. That means $c(u, j) \in S_{ui}$ and this contradicts $Fact$ 3, since we are assuming that the color exchange is proper. Thus $\{\lambda, \xi\} \neq \{c(u, i), c(u, j)\}$.

Now let P be the (λ,ξ,ab) critical path with respect to the coloring c. Without loss of generality assume that $\gamma = c(u, i) \in \{\lambda, \xi\}$. Since vertex u is contained in path P, by the maximality of the path P, it should contain the edge ui since $c(u, i) = \gamma \in \{\lambda, \xi\}$. Let us assume without loss of generality that path P starts at vertex a and reaches vertex i before it reaches vertex u. Now after the color exchange with respect to the edges ui and uj, i.e., with respect to the coloring c', there will not be any edge adjacent to vertex i that is colored γ. So if any (λ,ξ) maximal bichromatic path starts at vertex a, then it has to end at vertex i. Since $i \neq b$, by $Fact$ 1 we infer that the (λ,ξ,ab) critical path does not exist. ∎

3   Proof of Theorem 1

Proof. We prove the theorem by way of contradiction. Let G be a 2-degenerate graph with n vertices and m edges which is a minimum counter example for the theorem statement. Then the theorem is true for all 2-degenerate graphs with at most $m - 1$ edges. To prove the theorem for G, we may assume that G is connected. We may also assume that the minimum degree, $\delta(G) \geq 2$, since otherwise if there is a vertex v, with $degree(v) = 1$, we can easily extend the acyclic edge coloring of $G - e$ (where e is the edge incident on v) to G. Keeping the assumption that G is a minimum counter example in mind we will show that any partial coloring c of G should satisfy certain properties which in turn will lead to a contradiction.

Selection of the Primary Pivot: Let $W_0 = \{z \in V(G) \mid degree_G(z) = 2\}$. Since G is 2-degenerate $W_0 \neq \emptyset$. We may assume that $V - W_0 \neq \emptyset$ because otherwise, G is a cycle and it is easy to see that it is $\Delta + 1 = 3$ acyclically edge colorable. Thus $\Delta(G) \geq 3$. Let $G' = G[V - W_0]$ and $W_1 = \{z \in V(G') \mid degree_{G'}(z) \leq 2\}$. By the definition of 2-degeneracy there exists at least one vertex of degree at most 2 in G' and thus $W_1 \neq \emptyset$.

Let $V' = V(G')$. If $V' - W_1 \neq \emptyset$, then there exists at least one vertex of degree at most 2 in $G'[V' - W_1]$. Let $G'' = G[V' - W_1]$ and $W_2 = \{z \in V(G'') \mid degree_{G''}(z) \leq 2\}$. Let $q \in W_2$. Clearly $N_G(q) \cap W_1 \neq \emptyset$ and let $x \in N_G(q) \cap W_1$. On the other hand if $V' - W_1 = \emptyset$, then let $x \in W_1$. We call x the $Primary$ $Pivot$, since x plays an important role in our proof. Let $N'_G(x) = N_G(x) \cap W_0$ and $N''_G(x) = N_G(x) - N'_G(x) = N_{G'}(x)$. Since $x \in W_1$, it is easy to see that $|N''_G(x)| \leq 2$ and $|N'_G(x)| \geq 1$.

Let $N'_G(x) = \{y_1, y_2, \ldots, y_t\}$. Also $\forall y_i$, let $N_G(y_i) = \{x, y'_i\}$ (See $figure$ 1). $\forall y_i$, let G_i denote the graph obtained by removing the edge (x, y_i) from the graph G. Let $N'_{G_i}(x) = N'_G(x) - \{y_i\}$ and $N''_{G_i}(x) = N_{G_i}(x) - N'_{G_i}(x)$. By induction on the number of edges, graph G_i is $\Delta + 1$ acyclically edge colorable. Let c_i be a valid coloring of G_i and thus a partial coloring of G. We denote the set of colors by $C = \{1, 2, \ldots, \Delta + 1\}$.
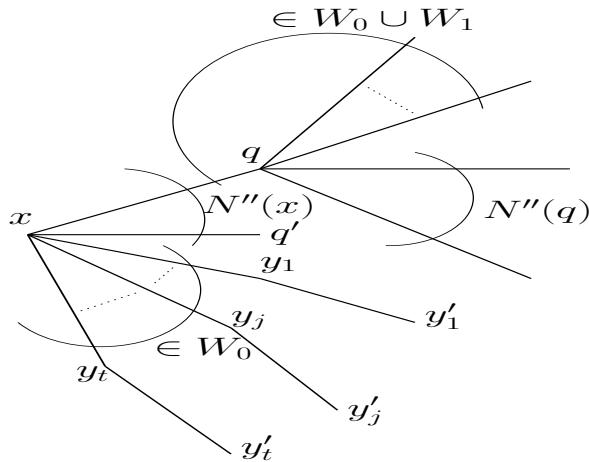

Figure 1: Vertex x and its neighbours

Comment: Note that the figures given in this paper are only for providing visual aid for the reader. They do not capture all possible configurations.

3.1 Properties of any valid coloring c_i of G_i

Let $F_x(c_i) = \{c_i(x, z) | z \in N_{G_i}(x)\}$. Let $F'_x(c_i) = \{c_i(x, z) | z \in N'_{G_i}(x)\}$ and $F''_x(c_i) = \{c_i(x, z) | z \in N''_{G_i}(x)\}$. Note that $F_x(c_i)$ is the disjoint union of $F'_x(c_i)$ and $F''_x(c_i)$ and also $|F''_x(c_i)| \leq 2$.

Lemma 2. *With respect to any valid coloring c_i of G_i, $c_i(y_i, y'_i) \in F''_x(c_i)$.*

Proof. It is easy to see that $c_i(y_i, y'_i) \in F_x(c_i)$. Otherwise all the candidate colors are valid for the edge xy_i, since any cycle involving the edge xy_i will contain the edge $y_i y'_i$ as well as an edge incident on x in G_i and thus the cycle will have at least 3 colors. Now if $c_i(y_i, y'_i) \in F'_x(c_i)$ we have $|F_x(c_i) \cup \{c_i(y_i, y'_i)\}| \leq \Delta - 1$. Thus there are at least two *candidate colors* for the edge xy_i. Let $y_j \in N'_{G_i}(x)$ be the vertex such that $c_i(y_i, y'_i) = c_i(x, y_j)$. When we color edge xy_i there is a possibility of a bichromatic cycle only if we assign $c_i(y_j, y'_j)$ to the edge xy_i since $degree_{G_i}(y_j) = 2$. But since we have at least two *candidate colors* for edge xy_i, this situation can easily be avoided. We infer that $c_i(y_i, y'_i) \in F''_x(c_i)$. ∎

Lemma 3. *With respect to any valid coloring c_i of G_i, $|F''_x(c_i)| = 2$*

Proof. Suppose not. Then $|F''_x(c_i)| \leq 1$. Since $|F_{y'_i}(c_i)| \leq \Delta$, we have at least one *candidate color* for the edge $y_i y'_i$. Note that any *candidate color*, is $valid$ for the edge $y_i y'_i$ in G_i since y_i is a pendant vertex in G_i. Let c'_i be the valid coloring obtained by recoloring the edge $y_i y'_i$ with a candidate color. By $Lemma\ 2$, we have $c_i(y_i, y'_i) \in F''_x(c_i)$. Clearly since $|F''_x(c_i)| \leq 1$ and $c'_i(y_i, y'_i) \neq c_i(y_i, y'_i)$, we can infer that $c'_i(y_i, y'_i) \notin F''_x(c_i)$, a contradiction to $Lemma\ 2$. ∎

An immediate consequence of $Lemma\ 3$ is that $|N''_G(x)| = 2$. Moreover by the way we have selected vertex x at least one of them should belong to $W_1 \cup W_2$. We make the following assumption:

Assumption 1. *With respect to any valid coloring c_i of G_i, without loss of generality let $F''_x(c_i) = \{1, 2\}$ and $N''_G(x) = \{q, q'\}$. Thus $\{c_i(x, q), c_i(x, q')\} = \{1, 2\}$. Also without loss of generality we assume that $q \in W_2 \cup W_1$ (see $figure\ 1$).*

Lemma 4. *With respect to any valid coloring c_i of G_i, colors $1, 2 \notin S_{y_i y'_i}$.*

Proof. Since $|F_{y'_i}(c_i)| \leq \Delta$, we have at least one *candidate color* $\gamma \neq c_i(y_i, y'_i)$ for the edge $y_i y'_i$. Note that γ is $valid$ for the edge $y_i y'_i$ in G_i since y_i is a pendant vertex in G_i. Let c'_i be the valid coloring obtained by recoloring the edge $y_i y'_i$ with γ. Now since c_i as well as c'_i are valid, by $Lemma\ 2$, we have $\{c_i(y_i, y'_i), c'_i(y_i, y'_i)\} = F''_x(c_i) = \{1, 2\}$ (by $Assumption$ 1). Since $c_i(y_i, y'_i) \notin S_{y_i y'_i}$ and $c'_i(y_i, y'_i) \notin S_{y_i y'_i}$, we have $1, 2 \notin S_{y_i y'_i}$. ∎

Let $C' = C - \{1, 2\}$. For each color $\gamma \in C'$, we define a graph $G_{i,\gamma}$ as below:

$$G_{i,\gamma} = \begin{cases} G_i & \text{if } \gamma \in C' - F'_x(c_i) \\ G_i - xy_a, \text{ where } c_i(x, y_a) = \gamma & \text{if } \gamma \in F'_x(c_i) \end{cases}$$

Also let $c_{i,\gamma}$ be the valid coloring of $G_{i,\gamma}$ derived from c_i of G_i, that is by discarding the color of the edge xy_a, where y_a is the vertex such that $c_i(x, y_a) = \gamma$. Also if $c_{i,\gamma}$ is a valid coloring of $G_{i,\gamma}$, then $c_{i,\gamma}$ is said to be derivable from c_1 if we can extend the coloring $c_{i,\gamma}$ of $G_{i,\gamma}$ to the coloring c_1 of G_1.

Lemma 5. *Let c_i be any valid coloring of G_i. With respect to coloring $c_{i,\gamma}$ of $G_{i,\gamma}$, $\forall \gamma \in C' - F'_x(c_i)$, $\exists (\mu, \gamma, xy_i)$ critical path, where $\mu = c_i(y_i, y'_i)$.*

Proof. Recall that when $\gamma \in C' - F'_x(c_i)$, we have $G_{i,\gamma} = G_i$ and hence $c_{i,\gamma} = c_i$. Suppose if there is no (μ, γ, xy_i) critical path, where $\gamma \in C' - F'_x(c_i)$, then by $Fact\ 2$ color γ is valid for the edge xy_i. Thus we get a valid coloring of G, a contradiction. ∎

Lemma 6. *Let c_i be any valid coloring of G_i. With respect to coloring $c_{i,\gamma}$ of $G_{i,\gamma}$, $\forall \gamma \in C' - F'_x(c_i)$, $\exists (\nu, \gamma, x, y'_i)$ maximal bichromatic path, where $\{\nu\} = \{1, 2\} - \{c_i(y_i, y'_i)\}$.*

Proof. Recall that when $\gamma \in C' - F'_x(c_i)$, we have $G_{i,\gamma} = G_i$ and hence $c_{i,\gamma} = c_i$. Suppose if there is no (ν, γ, x, y'_i) maximal bichromatic path, where $\gamma \in C' - F'_x(c_i)$, then by $Lemma\ 4$, color ν is a candidate for the edge $y_i y'_i$. Now recolor the edge $y_i y'_i$ with color ν to get a valid coloring c'_i of G_i. Since by our assumption that there is no (ν, γ, x, y'_i) maximal bichromatic path with respect to $c_{i,\gamma} = c_i$, there cannot be any (ν, γ, xy_i) critical path with respect to the coloring c'_i, a contradiction to $Lemma\ 5$ (Note that the color μ discussed in Lemma 5 and assumption is same as $\nu = c'_i(y_i, y'_i)$ in c'_i). ∎

Assumption 2. *Since $|F_x(c_i)| \leq \Delta - 1$, we have $|C - F_x(c_i)| \geq 2$. Since $C - F_x(c_i) = C' - F'_x(c_i)$, we have $|C' - F'_x(c_i)| \geq 2$. Thus $degree_{G_i}(y'_i) \geq 3$ and hence $degree_G(y'_i) \geq 3$. Let $\alpha, \beta \in C' - F'_x(c_i)$.*

Lemma 7. *Let c_i be any valid coloring of G_i. With respect to coloring $c_{i,\gamma}$ of $G_{i,\gamma}$, $\forall \gamma \in F'_x(c_i)$, $\exists (\mu, \gamma, xy_i)$ critical path, where $\mu = c_i(y_i, y'_i)$.*

Proof. Let $c_i(x, y_j) = \gamma$, where $\gamma \in F'_x(c_i)$. Suppose if there is no (μ, γ, xy_i) critical path, then by $Fact\ 2$ color γ is valid for the edge xy_i with respect to the coloring $c_{i,\gamma}$. Color the edge xy_i with color γ to get a valid coloring d of $G - \{xy_j\}$. Now we will show that we can extend the coloring d of $G - \{xy_j\}$ to a valid coloring of the graph G by giving a valid color for the edge xy_j, leading to a contradiction of our assumption that G was a minimum counter example. We claim the following:

Claim 1. *With respect to the coloring d, either color α or β is valid for the edge xy_j (Recall that $\alpha, \beta \in C' - F'_x(c_i)$ by Assumption2)*

Proof. Without loss of generality, let $d(y_j, y'_j) = \eta$. Note that $\eta \neq \gamma = c_i(x, y_j)$. Now if,

1. $\eta \notin F_x(c_i)$. In view of Assumption 2, $\alpha, \beta \notin F_x(c_i)$. Noting that η cannot be equal to both α and β, without loss of generality, let $\eta \neq \alpha$. Then color the edge (x, y_j) with color α to get a proper coloring d'. If a bichromatic cycle gets formed, then it should contain the edge xy_j and also involve both the colors η and α since $degree_G(y_j) = 2$. But since $\eta \notin F_x(c_i)$, such a bichromatic cycle is not possible. Thus the coloring d' is valid.

2. $\eta \in \{1, 2\} = \{\mu, \nu\} = F''_x(c_i)$. Recolor the edge xy_j with color α to get a coloring d'. We claim that the coloring d' is valid. This is because if it is not valid, then there has to be a (α, η) bichromatic cycle containing the edge xy_j with respect to d'. This implies that there has to be a (η, α, xy_j) critical path with respect to the coloring d and hence with respect to the coloring $c_{i,\gamma}$ (Note that the coloring d is obtained from $c_{i,\gamma}$ just by giving the color γ to the edge xy_i and $\eta, \gamma \neq \alpha, \beta$).

   If $\eta = \mu$, this means that there was a $(\eta = \mu, \alpha, xy_j)$ critical path with respect to $c_{i,\gamma}$. But this is not possible by $Fact\ 1$ since there is already a (μ, α, xy_i) critical path with respect to $c_{i,\gamma}$ (by $Lemma\ 5$) and $y_i \neq y_j$.

   Thus $\eta = \nu$. This means that there has to be a $(\eta = \nu, \alpha, xy_j)$ critical path with respect to $c_{i,\gamma}$. But this is not possible by $Fact\ 1$ since there is already a (ν, α, x, y'_i) maximal bichromatic path with respect to $c_{i,\gamma}$ (by $Lemma\ 6$) and $y'_i \neq y_j$ ($y'_i \neq y_j$ since by $Assumption\ 2$, $degree_{G_i}(y'_i) \geq 3$. But $degree_{G_i}(y_j) = 2$). Thus there cannot be any bichromatic cycles with respect to the coloring d'. Thus the coloring d' is valid.

3. $\eta \in F'_x(c_i)$. Let $y_k \in N'_G(x)$ be such that $d(x, y_k) = \eta$. With respect to colors $\{\alpha, \beta\}$, without loss of generality let $d(y_k, y'_k) \neq \beta$. Recall that $d(y_j, y'_j) = \eta$. Now recolor the edge xy_j with color β to get a coloring d'. Now if a bichromatic cycle gets formed, then it should contain the edge xy_j and also involve both the colors η and β. Thus the bichromatic cycle should contain the edge xy_k. Since $degree_G(y_k) = 2$, the bichromatic cycle should contain the edge $y_k y'_k$. But by our assumption, $c_i(y_k, y'_k) \neq \beta$, a contradiction. Thus the coloring d' is valid.

Hence either color α or β is valid for the edge xy_j.

$\square$

Thus we have a valid coloring (i.e, d') for the graph G, a contradiction.

$\blacksquare$

Lemma 8. *Let c_i be any valid coloring of G_i. With respect to coloring $c_{i,\gamma}$ of $G_{i,\gamma}$, $\forall \gamma \in F'_x(c_i)$, $\exists (\nu, \gamma, x, y'_i)$ maximal bichromatic path, where $\{\nu\} = \{1, 2\} - \{c_i(y_i, y'_i)\}$.*

Proof. Suppose if there is no (ν, γ, x, y'_i) maximal bichromatic path, where $\gamma \in F'_x(c_i)$, then by *Lemma* 4, color ν is a candidate for the edge $y_i y'_i$. Now recolor the edge $y_i y'_i$ with color ν to get a valid coloring $c'_{i,\gamma}$ of G_i. Since by our assumption that there is no (ν, γ, x, y'_i) maximal bichromatic path with respect to $c_{i,\gamma}$, there cannot be any (ν, γ, xy_i) critical path with respect to the coloring $c'_{i,\gamma}$, a contradiction to *Lemma* 7 (Note that the color μ discussed in Lemma 7 and assumption is same as $\nu = c'_{i,\gamma}(y_i, y'_i)$ in $c'_{i,\gamma}$).

$\blacksquare$

Critical Path Property: In the rest of the paper we will have to repeatedly use the properties (namely the presence of (μ, γ, xy_i) critical path in $G_{i,\gamma}$, where $\mu = c_{i,\gamma}(y_i, y'_i)$) described by *Lemma* 5 and *Lemma* 7. Therefore we will name these properties as the *Critical Path Property* of the graph $G_{i,\gamma}$

If c_i is any valid coloring of G_i, then in $G_{i,\gamma}$, $\forall \gamma \in C'$, by *Critical Path Property* (i.e., *Lemma* 5 or *Lemma* 7) there exists a (μ, γ, xy_i) critical path and by *Lemma* 6 and *Lemma* 8 there exists a (ν, γ, x, y'_i) maximal bichromatic path, where $\mu = c_i(y_i, y'_i)$ and $\{\nu\} = F''_x(c_i) - \{\mu\}$. Recall that $|S_{ab}| \leq \Delta - 1$ for any $ab \in E$. As an immidiate consequence we have,

$$S_{xq} = S_{xq'} = S_{y_i y'_i} = C - \{1, 2\} = C'. \tag{1}$$

In view of (1), we have

$$|S_{xq}| = |S_{xq'}| = |S_{y_i y'_i}| = |C'| = \Delta - 1. \tag{2}$$

Lemma 9. *Let c_i be any valid coloring of G_i. Let $\mu = c_i(y_i, y'_i) \in \{1, 2\}$. Also let $y_j \in N'_G(x) - \{y_i\}$. Then $\forall \gamma \in C'$, the (μ, γ, xy_i) critical path in $G_{i,\gamma}$ does not contain the vertex y_j.*

Proof. Suppose there exists a (μ, γ, xy_i) critical path that contains the vertex y_j, then y_j cannot be an end vertex as $y_i \neq y_j$. Thus y_j is an internal vertex. Now since $degree_G(y_j) = 2$, the (μ, γ, xy_i) critical path should contain the edge xy_j as well. But the (μ, γ, xy_i) critical path ends at vertex x with color μ which implies $c_i(x, y_j) = \mu$, a contradiction since $c_i(x, y_j) \notin \{1, 2\} = \{\mu, \nu\}$. $\blacksquare$

Lemma 10. *Let c_i be any valid coloring of G_i and let $u \in \{q, q'\}$. Let $\mu = c_i(y_i, y'_i) = c_i(x, u) \in \{1, 2\}$ and $\nu = \{1, 2\} - \{\mu\}$. Then $\forall \gamma \in C'$, the (μ, γ, xy_i) critical path in $G_{i,\gamma}$ has length at least five.*

Proof. Suppose not. Then the (μ, γ, xy_i) critical path has length three which implies that the vertices in the critical path are x, u, y'_i, y_i in that order. Thus $\mu \in F_u(c_i)$ and $F_u(c_i) = S_{xu} \cup \{\mu\}$. Now change the color of the edge $y'_i y_i$ to ν. It is proper since by 1, we have $\{1, 2\} = \{\mu, \nu\} \notin S_{y_i y'_i}$. It is valid since y_i is a pendant vertex in $G_{i,\gamma}$. Now in view of *Critical Path Property* (i.e., *Lemma* 5 or *Lemma* 7) there has to be a (ν, γ, xy_i) critical path that passes through the vertex y'_i with respect to this new coloring. Since $c_{i,\gamma}(u, y'_i) = \gamma$, this (ν, γ, xy_i) critical path should contain vertex u as an internal vertex, which implies that color $\nu \in F_u(c_i)$. Recalling that $F_u(c_i) = S_{xu} \cup \{\mu\}$, we have $\nu \in S_{xu}$, a contradiction in view of (1). Thus the (μ, γ, xy_i) critical path has length at least five with respect to the coloring $c_{i,\gamma}$ of $G_{i,\gamma}$. $\blacksquare$

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3.2 The structure of the minimum counter example in the vicinity of the primary pivot, x

Lemma 11. *The minimum counter example G satisfies the following properties,*

(a) $\forall u, v \in N_G(x)$, $(u, v) \notin E(G)$.

(b) $\forall y_i \in N'_G(x)$ *and* $\forall v \in N_G(x)$, *we have* $(v, y'_i) \notin E(G)$.

Proof. To prove (a) we consider the following cases:
case 1.1: $u, v \in N'_G(x)$
Let $u = y_k$ and $v = y_j$. Now if $u \in N_G(v)$, then $u = y'_j$. Recalling that $\Delta(G) \geq 3$, in view of (2), we have $degree_G(u) = degree_G(y'_j) \geq 3$. But $degree_G(u) = degree_G(y_k) = 2$, a contradiction.

case 1.2: $u, v \in N''_G(x)$
Then we need to show that $q' \notin N_G(q)$. To see this consider the coloring c_i of graph G_i. We know that $\{c_i(x, q), c_i(x, q')\} = \{\mu, \nu\}$. Without loss of generality let $c_i(x, q) = c_i(y_i, y'_i) = \mu$. Note that by (2), we have $S_{xq} = C'$. If $q' \in N_G(q)$, then $c_i(q, q') \in C'$. Let $c_i(q, q') = \gamma \notin \{\mu, \nu\}$. Now in $G_{i,\gamma}$, the (μ, γ) maximal bichromatic path that starts at vertex x contains only edges xq and qq' since $\mu \notin F_{q'}(c_i)$ (by (2)). Thus by $Fact\ 1$, there cannot be a (μ, γ, xy_i) critical path in $G_{i,\gamma}$, a contradiction to $Critical\ Path\ Property$ (i.e., $Lemma\ 5$ or $Lemma\ 7$). Thus $q' \notin N_G(q)$.

case 1.3: $u \in N''_G(x)$ and $v \in N'_G(x)$
Let $v = y_i$. Then we have to show that $y'_i \notin N''_G(x) = \{q, q'\}$. To see this consider the coloring c_i of graph G_i. Recall that $\{c_i(x, q), c_i(x, q')\} = \{\mu, \nu\}$. Without loss of generality let $c_i(x, q) = c_i(y_i, y'_i) = \mu$. Now if $y'_i = q$, then we have $c(q, y_i) = c_i(y'_i, y_i) = \mu$, a contradiciton since $c(x, q) = \mu$. On the other hand if $y'_i = q'$, then $c(q', y_i) = c_i(y'_i, y_i) = \mu$. This means that $\mu \in S_{xq'}$, a contradiction in view of (1). Thus $y'_i \neq q, q'$.

Thus $\forall u, v \in N_G(x)$, we have $(u, v) \notin E(G)$

To prove (b) we consider the following cases:
case 2.1: $v \in N'_G(x)$
Let $v = y_j \in N'_G(x)$. If $(v, y'_i) = (y_j, y'_i) \in E(G)$, then $y'_i = y'_j$. Consider the coloring c_j of graph G_j. Let $c_j(y_j, y'_j) = \mu$. Recall that by (2), we have $S_{y_j y'_j} = C'$. If $y'_i = y'_j$, then $c_j(y'_j, y_i) \in C'$. Let $c_j(y'_j, y_i) = \gamma$. Now in $G_{j,\gamma}$, the (μ, γ) maximal bichromatic path that starts at vertex y_j contains only edges $y_j y'_j$, $y'_j y_i$ and thus ends at vertex y_i since $\mu \notin F_{y_i}(c_j)$. This is because $N_{G_{j,\gamma}}(y_i) = \{y'_j, x\}$ and we have $c_j(y'_j, y_i) = \gamma$ and $c_j(x, y_i) \neq \mu$ (since by $Assumption\ 1$, $\mu \in c_j(x, q), c_j(x, q')$). Thus by $Fact\ 1$, there cannot be a (μ, γ, xy_j) critical path in $G_{j,\gamma}$, a contradiction to $Critical\ Path\ Property$ (i.e., $Lemma\ 5$ or $Lemma\ 7$). Thus $y'_j \neq y'_i$.

case 2.2: $v \in N''_G(x) = \{q, q'\}$
Then we have to show that $y'_i \notin N_G(q) \cup N_G(q')$. To see this consider the coloring c_i of graph G_i. Recall that $\{c_i(x, q), c_i(x, q')\} = \{\mu, \nu\}$. Without loss of generality let $c_i(x, q) = c_i(y_i, y'_i) = \mu$. Suppose $y'_i \in N_G(q)$, then we have $c(y'_i, q) \in S_{xq}$. Thus by (1), we have $c_i(y'_i, q) \neq \nu$. Now there exists a $(\mu, c_i(y'_i, q) \neq \nu, xy_i)$ critical path of length 3, a contradiction to $Lemma\ 10$. Now if $y'_i \in N_G(q')$, then we recolor the edge $y_i y'_i$ with color ν to get a valid coloring c'_i. Now there exists a $(\nu, c_i(y'_i, q'), xy_i)$ critical path of length 3, a contradiction to $Lemma\ 10$. Thus $y'_i \notin N_G(q) \cup N_G(q')$.

Thus $\forall y_i \in N'_G(x)$ and $\forall v \in N_G(x)$, we have $(v, y'_i) \notin E(G)$.

∎

3.3 Modification of valid coloring c_1 of G_1 to get valid coloring c_j of G_j

Assumption 3. *Let c_1 be a valid coloring of G_1 and without loss of generality let $c_1(x, q) = 1$, $c_1(x, q') = 2$ and $c_1(y_1, y'_1) = \mu = 1$.*

Remark: In view of $Assumption$ 3, the $Critical\ Path\ Property$ with respect to the coloring c_1 of G_1 reads as follows: With respect to the coloring $c_{1,\gamma}$, there exists a $(1,\gamma,xy_1)$ critical path, for all $\gamma \in C'$.

Let f_1 be the coloring of G_1 obtained from c_1 by exchanging the colors of the edges xq and xq'. Also for $\gamma \in C'$, we define the coloring $f_{1,\gamma}$ as the coloring obtained from $c_{1,\gamma}$ by exchanging the colors with respect to the edges xq and xq'. Note that $f_{1,\gamma}$ can be obtained from f_1 just by discarding the γ colored edge incident on vertex x for $\gamma \in F'_x(f_1)$.

Claim 2. *The coloring f_1 is proper but is not valid.*

Proof. The coloring f_1 is proper since in view of (1), $2 \notin S_{xq}$ and $1 \notin S_{xq'}$. Suppose the coloring f_1 is valid. Let γ be a candidate color for the edge xy_1. Clearly $\gamma \in C - F_x(f_1)$. Now since f_1 is proper, taking $u = x$, $i = q$, $j = q'$, $ab = xy_1$, $\lambda = 1$ and $\xi = \gamma$, $Lemma$ 1 can be applied. There existed a $(1,\gamma,xy_1)$ critical path with respect to coloring c_1. By $Lemma$ 1, we infer that there cannot be any $(1,\gamma,xy_1)$ critical path with respect to the coloring f_1. Thus by Fact 2, candidate color γ is valid for the edge xy_1. Thus we have obtained a valid coloring for the minimum counter example G, a contradiction. $\square$

By $Claim$ 2, there exist bichromatic cycles with respect to the coloring f_1. It is clear that each bichromatic cycle with respect to f_1 has to contain either the edge xq or xq' since we have changed only the colors of the edges xq and xq' to get the coloring f_1 from c_1. Thus each such bichromatic cycle should be either a $(1,\gamma)$ bichromatic cycle or a $(2,\gamma)$ bichromatic cycle. Note that each of these bichromatic cycles should pass through the vertex x. Moreover observe that there cannot be any $(1,2)$ bichromatic cycle since color $1 \notin S_{xq}$ with respect to f_1 in view of (1). Thus $\gamma \in F'_x(f_1)$. From this we infer that $|F'_x(f_1)| \geq 1$. Recalling $Assumption$ 2, we have $|C - F_x(f_1)| \geq 2$. It follows that $|C'| \geq 3$. Thus we have,

$$\Delta(G) \geq degree_{G_1}(q) \geq |S_{xq}| + 1 \geq |C'| + 1 \geq 4. \tag{3}$$

Let

$$C_1 = C_1(f_1) = \{\gamma \in F'_x(c_1)|\ \exists (1,\gamma)\ \text{bichromatic cycle with respect to coloring } f_1\}.$$

$$C_2 = C_2(f_1) = \{\gamma \in F'_x(c_1)|\ \exists (2,\gamma)\ \text{bichromatic cycle with respect to coloring } f_1\}.$$

Note that from the discussion above, any bichromatic cycle with respect to the coloring f_1 contains a vertex $y_i \in N'_{G_1}(x)$. But $degree_{G_1}(y_i) = 2$ and therefore $|S_{xy_i}| = 1$. Thus S_{xy_i} contains exactly one of the color 1 or 2. Thus with a fixed color $\gamma \in C_1 \cup C_2$ there exists exactly one of $(1,\gamma)$ or $(2,\gamma)$ bichromatic cycle, which implies that the sets C_1 and C_2 cannot have any element in common (See $figure$ 2). Thus we have,

$$C_1 \cap C_2 = \emptyset. \tag{4}$$

Recall that in view of $Critical\ Path\ Property$ (i.e., $Lemma$ 5 or $Lemma$ 7), for a coloring $c_{1,\gamma}$ of $G_{1,\gamma}$, $\forall \gamma \in C'$, there exists a $(1,\gamma,xy_1)$ critical path. With respect to the new coloring $f_{1,\gamma}$, since the colors of only edges xq and xq' are changed, this path starts from y_1 and reaches the vertex q. But since color 1 is not present at vertex q with respect to the coloring $f_{1,\gamma}$, the bichromatic path ends at vertex q. Thus the $(1,\gamma,xy_1)$ critical path with respect to coloring $c_{1,\gamma}$ gets curtailed to $(\gamma,1,q,y_1)$ maximal bichromatic path with respect to $f_{1,\gamma}$. Also note that in view of $Lemma$ 10 the length of this $(\gamma,1,q,y_1)$ maximal bichromatic path is at least four. This is true for the coloring f_1 also i.e., there exists a $(\gamma,1,q,y_1)$ maximal bichromatic path with respect to f_1. To see this observe that f_1 is obtained from $f_{1,\gamma}$ by putting back the edge xy_a, where $c_1(x,y_a) = \gamma$. This cannot alter the $(\gamma,1,q,y_1)$ maximal bichromatic path since x does not belong to this path and also $y_a \neq y_1$. Also in view of Lemma 9, none of the above maximal bichromatic paths contain vertex y_j, $\forall y_j \in N'_G(x) - \{y_1\}$. Thus the coloring f_1 satisfies the following property which we name as $Property\ A$:

Property A: A partial coloring of G is said to satisfy $Property\ A$ iff $\forall \gamma \in C - \{1,2\}$, there exists a $(\gamma,1,q,y_1)$ maximal bichromatic path of length at least four. Moreover none of the above maximal bichromatic paths contain vertex x or vertex y_i, where $y_i \in N'_G(x) - \{y_1\}$.

Claim 3. *There exists a proper coloring f'_1 obtained from f_1 such that $\forall i \in \{1,2\}$, $|C_i| \leq 1$, where $C_i = C_i(f'_1)$. Moreover f'_1 satisfies $Property\ A$.*
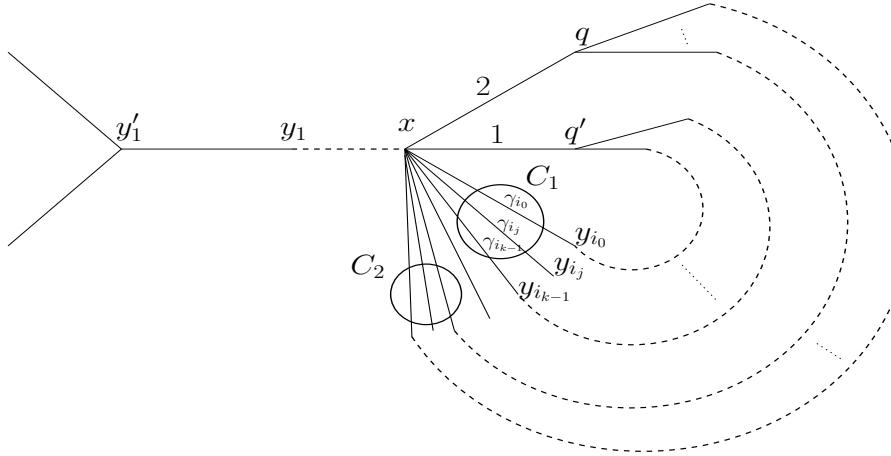
Figure 2: Bichromatic cycles of C_1 and C_2

Proof. If $|C_1| \leq 1$ and $|C_2| \leq 1$, then let $f'_1 = f_1$. If $|C_1| \leq 1$, then let $f''_1 = f_1$. Otherwise if $|C_1| \geq 2$, then let $C_1 = \{\gamma_{i_0}, \gamma_{i_1}, \ldots, \gamma_{i_{k-1}}\}$ and also let y_{i_j} be the vertex such that $f_1(x, y_{i_j}) = \gamma_{i_j}, \forall j \in \{0, 1, 2, \ldots k-1\}$ (see $Figure$ 2). Now let the coloring f''_1 be defined as $f''_1(x, y_{i_j}) = \gamma_{i_l}$, where $l = j + 1 (mod\ k)$, $\forall j \in \{0, 1, 2, \ldots k-1\}$ and $f''_1(e) = f_1(e)$ for all other edges. (*Note that we have only shifted the colors of the edges $xy_{i_0}, xy_{i_1}, \ldots, xy_{i_{k-1}}$ circularly. We call this procedure* **deranging of colors.**)

Note that we are changing only the colors of the edges xy_{i_j} for $j = 0, 1, 2, \ldots, k-1$. Also we are using only the colors $\gamma_{i_j} \in C_1$ for recoloring. Since with respect to the coloring f_1, $(1, \gamma_{i_j})$ bichromatic cycle passed through y_{i_j} and $degree_{G_1}(y_{i_j}) = 2$, we have $S_{xy_{i_j}} = \{1\}$. Thus the coloring f''_1 is proper.

Since for all $y_{i_j}, 0 \leq j \leq k-1$ we have $S_{xy_{i_j}} = \{1\}$ with respect to the coloring f''_1, it is clear that any new bichromatic cycle created (in the process of getting f''_1 from f'_1) has to be a $(1, \gamma)$ bichromatic cycles, where $\gamma \in C_1$.

We claim that the coloring f''_1 does not have any $(1, \gamma)$ bichromatic cycle for $\gamma \in C_1$. To see this consider a $\gamma \in C_1$, say γ_{i_1}. There existed a $(1, \gamma_{i_1})$ bichromatic cycle with respect to f_1. It contained the edge xy_{i_1}. Now with respect to f''_1 edge xy_{i_1} is colored with color γ_{i_2}. Thus the $(1, \gamma_{i_1})$ maximal bichromatic path which contains the vertex x has one end at vertex y_{i_1} since color γ_{i_1} is not present at the vertex y_{i_1} with respect to f''_1. Thus $(1, \gamma_{i_1})$ bichromatic cycle cannot exist with respect to the coloring f''_1. This argument works for all $\gamma \in C_1$ and thus for any color $\gamma \in C_1$, there is no $(1, \gamma)$ bichromatic cycle with respect to f''_1.

If $|C_2| \leq 1$, then $f'_1 = f''_1$. Otherwise if $|C_2| \geq 2$, by performing similar recoloring (now starting with f''_1) as we did to get rid of the $(1, \gamma)$ bichromatic cycles, we can get a coloring f'''_1 without any $(2, \gamma)$ bichromatic cycle. Now let $f'_1 = f'''_1$. Thus we get a coloring f'_1 from f_1 which has $|C_1| \leq 1$ and $|C_2| \leq 1$.

Note that we are changing only the colors of the edges xy_i, for $y_i \in N'_{G_1}(x)$. But the coloring f_1 satisfied $Property\ A$ and hence none of the $(\gamma, 1, q, y_1)$ maximal bichromatic paths , $\forall \gamma \in C - \{1, 2\}$, contained the vertex y_i or x. Thus these bichromatic paths have not been altered (i.e., neither $broken$ nor $extended$) by the recoloring to get f'_1 from f_1. Thus the coloring f'_1 satisfies $Property\ A$. □

Observation 1. *Note that the color of the edge $y_1 y'_1$ is unaltered in f'_1, i.e., $f'_1(y_1, y'_1) = f_1(y_1, y'_1) = c_1(y_1, y'_1) = 1$. Also only the colors of certain edges incident on the vertex y_i, where $y_i \in N'_G(x) - \{y_1\}$ are modified when we obtained f'_1 starting from c_1. (*This information is required later in the proof*).*

It is easy to see that f'_1 is proper but not valid. It is not valid because, if it is valid then since f'_1 satifies $Property\ A$, there are $(\gamma, 1, q, y_1)$ maximal bichromatic paths , $\forall \gamma \in C - \{1, 2\}$. Thus by $Fact\ 1$, for any $\theta \in C - F_x(f'_1)$, there cannot be a $(1, \theta, xy_1)$ critical path. Thus by $Fact\ 2$, color θ is valid for the edge xy_1. Thus we have a valid coloring for the graph G, a contradiciton. Thus f'_1 is not valid. It implies that at least one of C_1 or C_2 is nonempty. In the next lemma we further refine the proper coloring f'_1.

Lemma 12. *There exists a proper coloring h_1 of G_1 obtained from f_1' such that there is at most one bichromatic cycle. Moreover h_1 satisfies Property A.*

Proof. By $Claim$ 3, we have $|C_1| \leq 1$ and $|C_2| \leq 1$. If exactly one of C_1, C_2 is singleton, then let $h_1 = f_1'$. Otherwise we have $|C_1| = 1$ and $|C_2| = 1$.

Assumption 4. *Without loss of generality let $C_1 = \{\gamma\}$ and $C_2 = \{\theta\}$. Let $f_1'(x, y_j) = \gamma$ and $f_1'(x, y_k) = \theta$. Thus $f_1'(y_j, y_j') = 1$ and $f_1'(y_k, y_k') = 2$, since there are $(1, \gamma)$ and $(2, \theta)$ bichromatic cycles passing through the vertex x.*

Claim 4. *Color $2 \notin S_{y_j y_j'}$*

Proof. Suppose not, then $2 \in S_{y_j y_j'}$. Since there is a $(1, \gamma)$ bichromatic cycle passing through y_j', the colors 1 and γ are present at y_j'. It follows that there exists $\eta \in C - \{1, 2, \gamma\}$ missing at y_j'. Now recolor edge $y_j y_j'$ with color η to get a coloring f_1''. If the color η is valid for the edge $y_j y_j'$, then let $h_1 = f_1''$ and we are done as the situation reduces to having only one bichromatic cycle (i.e.,$|C_2| = 1$ and $|C_1| = 0$). If the color η is not valid for the edge $y_j y_j'$, then there has to be a (γ, η) bichromatic cycle that passes through vertex x. Let $f_1''(x, y_l) = \eta$. Since $degree_G(y_l) = 2$, we have $S_{xy_l} = \{f_1''(y_l, y_l')\} = \{\gamma\}$. Recall that by $Assumption$ 2, $\alpha \in C' - F_x'(c_1)$ and thus $\alpha \in C' - F_x'(f_1'')$. Clearly $\alpha \neq \eta$. Recolor the edge xy_j with color α to get a coloring f_1'''. Note that the color α is valid for the edge xy_j because if there is a (α, η) bichromatic cycle, then it implies that $S_{xy_l} = \{\alpha\}$. But we know that $S_{xy_l} = \{\gamma\}$, a contradiction. Thus let $h_1 = f_1'''$ and the situation reduces to having only one bichromatic cycle (i.e., $|C_2| = 1$ and $|C_1| = 0$) □

In view of $Claim$ 4, color 2 is a candidate for the edge $y_j y_j'$. Recolor edge $y_j y_j'$ with color 2 to get a coloring f_1''. If the color 2 is valid for the edge $y_j y_j'$, then let $h_1 = f_1''$ and the situation reduces to having only one bichromatic cycle (i.e., $|C_2| = 1$ and $|C_1| = 0$). If the color 2 is not valid for the edge $y_j y_j'$, then there has to be a $(\gamma, 2)$ bichromatic cycle created due to the recoloring, thereby reducing the situation to $|C_2| = 2$ and $|C_1| = 0$. Now we can recolor the graph using the procedure similar to that in the proof of $Claim$ 3 (i.e., derangement of colors in C_2) to get a valid coloring h_1 without any bichromatic cycles.

The coloring f_1' satisfied $Property$ A and hence none of the $(\gamma, 1, q, y_1)$ maximal bichromatic paths , $\forall \gamma \in C - \{1, 2\}$, contained the vertex y_j. Thus none of the $(\gamma, 1, q, y_1)$ maximal bichromatic paths will be $broken$ or $curtailed$ in the process of getting h_1 from f_1'. This is because we are changing only the colors of the edges incident on the vertex y_j or y_k and if a $(\gamma, 1, q, y_1)$ maximal bichromatic path gets $broken$ or $curtailed$, it means that the vertex y_j or y_k was contained in those maximal bichromatic path, a contradiction to $Property$ A of f_1' since $y_j \in N_G'(x) - \{y_1\}$. On the other hand, if any of these paths gets extended, then vertex $y_j' \in \{y_1, q\}$. But in view of $Lemma$ 11 (part (a)) this is not possible. Thus the $(\gamma, 1, q, y_1)$ maximal bichromatic paths have not been extended. Thus these bichromatic paths have not been altered by the recolorings to get h_1 from f_1'. Thus the coloring h_1 satisfies $Property$ A. ■

Observation 2. *Note that the color of the edge $y_1 y_1'$ is unaltered in h_1, i.e., $h_1(y_1, y_1') = f_1'(y_1, y_1') = 1$ (by Observation 1). Also only the colors of certain edges incident on the vertex y_i, where $y_i \in N_G'(x) - \{y_1\}$ are modified.*

It is easy to see that h_1 is proper but not valid. It is not valid because, if it is valid then since h_1 satifies $Property$ A, there are $(\gamma, 1, q, y_1)$ maximal bichromatic paths , $\forall \gamma \in C - \{1, 2\}$. Thus by $Fact$ 1, for any $\theta \in C - F_x(h_1)$, there cannot be a $(1, \theta, xy_1)$ critical path. Thus by $Fact$ 2, color θ is valid for the edge xy_1. Thus we have a valid coloring for the graph G, a contradiciton. Thus h_1 is not valid. Then in view of $Lemma$ 12, we make the following assumption:

Assumption 5. *Without loss of generality let the only bichromatic cycle in the coloring h_1 of G_1 pass through the vertex y_j, $j \neq 1$. Also let $h_1(x, y_j) = \rho$.*

We get a coloring c_j of G_j from h_1 of G_1 by:

1. Removing the edge xy_j.

2. Adding the edge xy_1 and coloring it with the color $h_1(x, y_j) = \rho$.

Note that the coloring c_j is proper since $\rho \neq c_j(y_1, y_1') = h_1(y_1, y_1') = 1$ (by *Observation* 2) and $\rho \notin S_{y_1 x}(c_j)$ (by the definition of c_j). Note that by removing the edge xy_j we have broken the only bichromatic cycle that existed with respect to h_1. The coloring c_j is valid because if there is a bichromatic cycle in G_j with respect to c_j then it should contain the edge xy_1 and thus it should be a $(1, \rho)$ bichromatic cycle since $c_j(x, y_1) = h_1(x, y_j) = \rho$ and $c_j(y_1, y_1') = 1$. Suppose there exists a $(1, \rho)$ bichromatic cycle in G_j with respect to c_j, then by *Fact* 2 there must have been a $(1, \rho, xy_1)$ critical path with respect to h_1. But by *Lemma* 12, the coloring h_1 satisfies *Property A* and thus there was a $(\rho, 1, q, y_1)$ maximal bichromatic path implying by *Fact* 1 that there cannot be a $(1, \rho, xy_1)$ critical path with respect to h_1, a contradiction. It follows that the coloring c_j of G_j is acyclic. Therefore all the Lemmas in previous sections are applicable to the coloring c_j also.

Now we may assume that $c_j(y_j, y_j') = 2$ because if $c_j(y_j, y_j') = 1$, then we can change the color of the edge $y_j y_j'$ to 2 without altering the validity of the coloring since y_j is a pendant vertex in G_j. Thus we make the following assumption:

Assumption 6. *Without loss of generality let $c_j(y_j, y_j') = 2$. Also recall that $c_j(x, q) = 2$ and $c_j(x, q') = 1$.*

Remark: In view of *Assumption* 6, the *Critical Path Property* with respect to the coloring c_j of G_j reads as follows: With respect to the coloring $c_{j,\gamma}$, there exists a $(2, \gamma, xy_j)$ critical path, for all $\gamma \in C'$. The reader may contrast the *Critical Path Property* of c_j with that of c_1 (See remark after *Assumption* 3). This correspondence is very important for the proof.

Observation 3. *Note that $c_j(x, q) = 2$, $c_j(x, q') = 1$, $c_j(y_1, y_1') = 1$, $c_j(y_j, y_j') = 2$ and $c_j(x, y_1) = \rho \notin \{1, 2\}$. Also if e is an edge such that none of its end points is x or y_i, where $y_i \in N_G'(x)$, we have $c_j(e) = c_1(e)$.*

Lemma 13. *Coloring $c_{j,\gamma}$ of $G_{j,\gamma}$ satisfies Property A.*

Proof. We consider the following cases:
case 1: $\gamma \in C' - \rho$
Recall that the coloring h_1 satisfied *Property A*. In getting c_j from h_1, we have only colored the edge xy_1 with color ρ and have discarded the edge xy_j. Thus $\forall \gamma \in C' - \{\rho\}$, there exists a $(\gamma, 1, q, y_1)$ maximal bichromatic path in c_j also. Noting that by *Property A*, the maximal bichromatic path does not contain vertex x or y_i, where $\forall y_i \in N_G'(x) - \{y_1\}$, we infer that even in $G_{j,\gamma}$ the $(\gamma, 1, q, y_1)$ maximal bichromatic path is unaltered.
case 2: $\gamma = \rho$
Then $G_{j,\rho}$ is the graph obtained by removing the edge xy_1 from G_j since $c_j(x, y_1) = \rho$. Recall that with respect to the coloring h_1 we have a $(\rho, 1, q, y_1)$ maximal bichromatic path. Removal of edge xy_1 from G_1 cannot alter this path since h_1 satisfies *Property A* and thus edge xy_1 is not in the path. Now the graph obtained is nothing but the graph $G_{j,\rho}$ with respect to the coloring $G_{j,\rho}$. Thus $G_{j,\rho}$ saitsfies *Property A*. ∎

Property B: Let $c_{1,\eta}$ be a partial coloring of $G_{1,\eta}$, for $\eta \in C - \{1, 2\}$. Then $c_{1,\eta}$ is said to satisfy *Property B* iff $\forall \gamma \in C - \{1, 2\}$, there exists a $(\gamma, 2)$ maximal bichromatic path which starts at vertex q and involves the vertex y_j'. Also the length of the segment of this bichromatic path between the vertices q and y_j' is at least three. Moreover in none of the above maximal bichromatic paths the segment between the vertices q and y_j' contains vertex x or vertex y_i, where $y_i \in N_G'(x)$.

Lemma 14. *Coloring $c_{1,\eta}$ of $G_{1,\eta}$ satisfies Property B, for $\eta \in C - \{1, 2\}$.*

Proof. By *Critical Path Property* (i.e., *Lemma* 5 or *Lemma* 7) and *Lemma* 10, $\forall \gamma \in C'$, there exists a $(2, \gamma, x, y_j)$ critical path of length at least five in $G_{j,\gamma}$. Also by *Lemma* 9, these critical paths do not contain vertex y_i, $\forall y_i \in N_G'(x) - \{y_j\}$. Recall that we obtained c_j from c_1 by a series of recolorings. How will the above mentioned critical paths change if we undo all these recolorings and get back c_1? Note that in the process of obtaining coloring c_j from c_1, we have only changed the colors incident on the vertices y_i, where $y_i \in N_G'(x)$ and have exchanged the colors of the edges xq and xq' (by *Observation* 3). Thus only the colors of edge xq and possiblly edge $y_j y_j'$ of these critical paths will get modified when we undo the recolorings. The reader may recall that the first step in getting c_j from c_1 was to exchange the colors of edges xq and xq'. It follows that with respect to a coloring $c_{1,\eta}$, there exists a $(\gamma, 2)$ maximal bichromatic path which *starts at* vertex q and involves the vertex y_j'. It also follows that the length of the segment of the bichromatic path between the vertices q and y_j' is at least three. Moreover it is easy to see that none of the above maximal bichromatic paths the segment between the vertices q and y_j' contains vertex x or vertex y_i, where $y_i \in N_G'(x)$. ∎

3.4 Selection of secondary pivot p and properties of c_1 and c_j in the vicinity of p

Let $N'_G(q) = N_G(q) \cap (W_1 \cup W_0)$ and $N''_G(q) = N_G(q) - N'_G(q)$. Since $q \in W_2 \cup W_1$ (see *Assumption* 1) it is easy to see that $|N''_G(q)| \le 2$. Now recall that in view of (2) $degree_G(q) = \Delta$ and by (3), $\Delta \ge 4$. Thus we have $|N'_G(q)| \ge 2$.

Let $p \in N'_G(q)$ be such that $p \ne x$. In the rest of the proof, this vertex p will play a central role. Therefore we name it as the *Secondary Pivot*. Let $c_1(q,p) = \eta$. Note that $\eta \in C'$ by (1). Thus by *Critical Path Property* (i.e., *Lemma* 5 or *Lemma* 7), there exists a $(1, \eta, xy_1)$ critical path with respect to the coloring $c_{1,\eta}$ that passes through the vertex p and clearly qp is the second edge of this critical path. Recalling that this critical path has length at least five (by *Lemma* 10), we can infer that $p \ne y_1$ and $degree_{G_1}(p) \ge 2$. Now since $p \in W_1 \cup W_0$, there is at most one neighbour of p other than q which is not in W_0. If such a vertex exists let it be p'. Otherwise clearly $(N_G(p) \cap W_0) \ne \emptyset$ and let $p' \in N_G(p) \cap W_0$. Thus $N_G(p) - \{q, p'\} \subseteq W_0$. If $N_G(p) - \{q, p'\} \ne \emptyset$, let $N_G(p) - \{q, p'\} = \{z_1, z_2, \ldots, z_k\}$. Also $\forall z_i$, let $N_G(z_i) = \{p, z'_i\}$ (See *figure* 3) (At this point the reader may note that the primary pivot x and secondary pivot p are somewhat structurally similar).
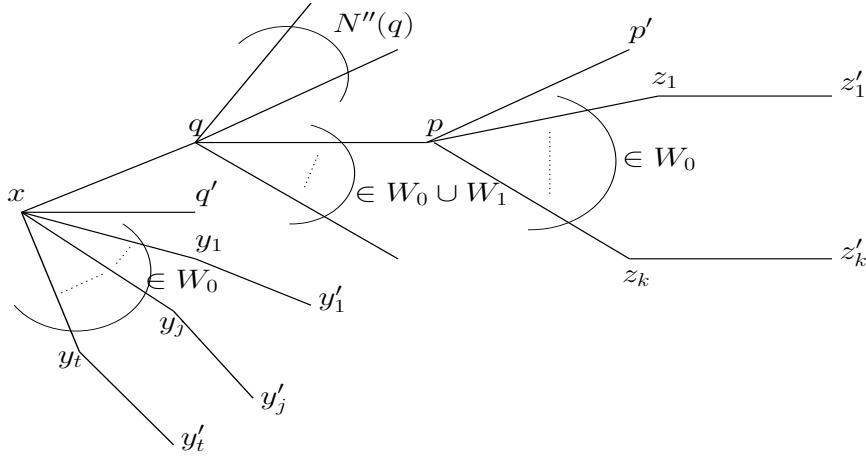
Figure 3: Vertex p and its neighbours

Lemma 15. $x, y_i \notin \{p, p', z_1, \ldots, z_k, z'_1, \ldots, z'_k\}$, *for* $y_i \in N'_G(x)$.

Proof. First note that $x \ne p$, by the definition of p. It is easy to see that $x \notin \{p', z_1, \ldots, z_k\}$, by part (a) of *Lemma* 11. Now $x \notin \{z'_1, \ldots, z'_k\}$ because otherwise z_i will be some y_i and hence $p = y'_i$, But now there is an edge between q and p, a contradiciton to part (b) of *Lemma* 11. Similarly from part (a) of *Lemma* 11, $y_i \ne p$ and from part (b) of *Lemma* 11, $y_i \notin \{p', z_1, \ldots, z_k\}$. Now if $y_i \in \{z'_1, \ldots, z'_k\}$, then since $x \ne z_i$, we have $y'_i = z_i$, a contradiciton since $degree_G(y'_i) = \Delta \ge 4$ (by (2)) and $degree_G(z_i) = 2$. Thus $x, y_i \notin \{p, p', z_1, \ldots, z_k, z'_1, \ldots, z'_k\}$, for $y_i \in N'_G(x)$. ∎

Lemma 16. $c_j(q,p) = c_1(q,p) = \eta$.

Proof. Recall that by *Observation* 3, only the edges incident on vertices x or y_i, where $y_i \in N'_G(x)$ are altered while obtaining coloring c_j from c_1. Now to show that $c_j(q,p) = c_1(q,p) = \eta$, its enough to verify that $q, p \notin \{x\} \cup N'_G(x)$. But this is obvious from part (a) of *Lemma* 11.

∎

Lemma 17. $\{1,2\} \subseteq S_{qp}(c_{1,\eta})$.

Proof. By *Lemma* 10, we know that $1 \in S_{qp}(c_{1,\eta})$ since qp is only the second edge of the $(1, \eta, xy_1)$ critical path which is guaranteed to have length at least five with respect to the coloring $c_{1,\eta}$ of $G_{1,\eta}$. Now by *Lemma* 14, with respect to $c_{1,\eta}$ there exists a $(\eta, 2)$ maximal bichromatic path which starts at vertex q and contains vertex y'_j. Moreover the segment of the bichromatic path between the vertices q and y'_j is of length at least three with respect to $c_{1,\eta}$. Since this $(\eta, 2)$ maximal bichromatic path starts with edge qp colored η, we infer that $2 \in S_{qp}(c_{1,\eta})$. Thus $\{1, 2\} \subseteq S_{qp}(c_{1,\eta})$.

∎

Remark: In view of *Lemma* 17, $degree_G \geq 3$. Therefore $p \notin W_0$. It follows that $p \in W_1$. It is interesting to note that p could have been selected as the *Primary Pivot* instead of x. The reader may want to reread the procedure for selecting the primary pivot given at the beginning of Section 3. With respect to this procedure vertex p is symmetric to vertex x and thus is an equally eligible candidate to be the primary pivot. It follows that the structure of the minimum counter example at the vicinity of p is symmetric to the structure at the vicinity of x. More specifically we have the following Lemma, corresponding to *Lemma* 11:

Lemma 18. *The minimum counter example G satisfies the following properties,*

(a) $\forall u, v \in N_G(p)$, $(u, v) \notin E(G)$.

(b) $\forall z_i \in N_G(p) - \{q, p'\}$ *and* $\forall v \in N_G(p)$, *we have* $(v, z'_i) \notin E(G)$.

This Lemma is not explicitly used in the proof, but we believe that this information will help the reader to visualize the situation better.

In view of *Lemma* 17 let e_1 and e_2 be the edges incident on p such that $c_{1,\eta}(e_1) = 1$ and $c_{1,\eta}(e_2) = 2$. Then we claim the following:

Lemma 19. $c_{j,\eta}(e_1) = 1$ *and* $c_{j,\eta}(e_2) = 2$.

Proof. Recall that by *Observation* 3, only the edges incident on vertices x or y_i, where $y_i \in N'_G(x)$ are altered while obtaining coloring c_j from c_1. Let $e_1 = (p, z_{i_1})$ and $e_2 = (p, z_{i_2})$. Now to show that $c_{j,\eta}(e_1) = 1$ and $c_{j,\eta}(e_2) = 2$, it is enough to verify that $p, z_{i_1}), z_{i_2}) \notin \{x\} \cup N'_G(x)$. But this true by *Lemma* 15.

∎

Lemma 20. $c_{1,\eta}(p, p') \in \{1, 2\}$ (*In other words, one of the edge e_1 or e_2 is pp'. By Lemma 19, this also implies that $c_{j,\eta}(p, p') = c_{1,\eta}(p, p') \in \{1, 2\}$*).

Proof. Suppose not. Then $e_1 \neq pp'$ and $e_2 \neq pp'$. Without loss of generality let $e_1 = (p, z_1)$ and $e_2 = (p, z_2)$. Thus $c_{1,\eta}(p, z_1) = 1$ and $c_{1,\eta}(p, z_2) = 2$. By *Lemma* 10 there exists a $(1, \eta, xy_1)$ critical path of length at least five with respect to $c_{1,\eta}$. This implies that $c_{1,\eta}(z_1, z'_1) = \eta$. Now by *Lemma* 14, with respect to $c_{1,\eta}$ there exists a $(\eta, 2)$ maximal bichromatic path which starts at vertex q and contains vertex y'_j. Moreover the segment of this bichromatic path between the vertices q and y'_j is of length at least three with respect to $c_{1,\eta}$. Since pz_1 is only the second edge of this path, we can infer that $c_{1,\eta}(z_2, z'_2) = \eta$.

Now with respect to the coloring $c_{1,\eta}$, we exchange the colors of the edges pz_1 and pz_2 to get a coloring $c'_{1,\eta}$.

Claim 5. *Coloring $c'_{1,\eta}$ is valid.*

Proof. Note that $c'_{1,\eta}$ is proper since $c_{1,\eta}(z_1, z'_1) = \eta$ and $c_{1,\eta}(z_2, z'_2) = \eta$. Now the coloring $c'_{1,\eta}$ is valid because otherwise there has to be a $(\eta, 1)$ or $(\eta, 2)$ bichromatic cycle since only the colors of the edges pz_1 and pz_2 are altered. Thus such a bichromatic cycle has to contain the edge qp since $c'_{1,\eta}(q, p) = \eta$. From (1), we can infer that color $2 \notin F_q(c'_{1,\eta})$. But if there exists a bichromatic cycle with respect to the coloring $c'_{1,\eta}$, it has to contain vertex q. From this we can infer that it has to be a $(\eta, 1)$ bichromatic cycle. This means that the cycle has to contain the vertex x since $c'_{1,\eta}(x, q) = 1$. But we know by defintion of $c_{1,\eta}$ that $\eta \notin F_x(c_{1,\eta}) = F_x(c'_{1,\eta})$. Thus there does not exist a $(\eta, 1)$ bichromatic cycle with respect to the coloring $c'_{1,\eta}$. We conclude that the coloring $c'_{1,\eta}$ of $G_{1,\eta}$ is valid. □

Claim 6. *With respect to the partial coloring $c'_{1,\eta}$, there does not exist any $(1, \eta, xy_1)$ critical path.*

Proof. Now since $c'_{1,\eta}$ is proper, taking $u = p$, $i = z_1$, $j = z_2$, $ab = xy_1$, $\lambda = 1$, $\xi = \eta$ and noting that $\{x, y_1\} \cap \{z_1, z_2\} = \emptyset$ (by *Lemma* 15), *Lemma* 1 can be applied. There existed a $(1, \eta, xy_1)$ critical path containing vertex p in coloring $c_{1,\eta}$. By *Lemma* 1, we infer that there cannot be any $(1, \eta, xy_1)$ critical path in the coloring $c'_{1,\eta}$. $\square$

Claim 7. *There exists a valid coloring c'_1 of G_1 such that the coloring $c'_{1,\eta}$ of $G_{1,\eta}$ is derivable from c'_1.*

Proof. It is enough to show that we can extend the coloring $c'_{1,\eta}$ of $G_{1,\eta}$ to a valid coloring c'_1 of G_1. If $\eta \in C - F_x(c_1)$, then by definition $G_{1,\eta} = G_1$ and thus $c'_1 = c'_{1,\eta}$. Otherwise let $y_k \in N'_G(x)$ be the vertex such that $c_1(x, y_k) = \eta$. Note that $k \neq 1$. Recall that $c_{1,\eta}$ is obtained by discarding the color on the edge xy_k. Thus it is enough to extend the coloring $c'_{1,\eta}$ to c'_1 by assigning an appropriate color to the edge xy_k.

Note that there exists a $(1, \alpha, xy_1)$ critical path with respect to $c_{1,\eta}$, for $\alpha \in C - F_x(c_1)$ (by *Lemma* 5). Clearly $\alpha \neq \eta$. We claim that the $(1, \alpha, xy_1)$ critical path exists even with respect to $c'_{1,\eta}$. To see this note that we have changed the colors of only edges pz_1 and pz_2 to get $c'_{1,\eta}$ from $c_{1,\eta}$. Note that by this exchange the $(1, \alpha, xy_1)$ critical path cannot be extended since $p, z_2 \notin \{x, y_1\}$ (by *Lemma* 15). Now if the $(1, \alpha, xy_1)$ critical path gets altered it means that this critical path contained the edge pz_1 (recall that $c_{1,\eta}(p, z_1) = 1$) and hence $c_{1,\eta}(z_1, z'_1) = \alpha$. But we know that $c_{1,\eta}(z_1, z'_1) = \eta$, a contradiction. Thus we have,

> With respect to the partial coloring $c'_{1,\eta}$, there exists a $(1, \alpha, xy_1)$ critical path, for $\alpha \notin F_x(c'_{1,\eta})$ and $\alpha \neq \eta$. $\hspace{2em}$ (5)

Now color the edge xy_k with color η to get a coloring d_1 of G_1. If d_1 is valid we are done and $c'_1 = d_1$. If it is not valid, then there has to be a bichromatic cycle containing the color η. Note that the coloring d_1 and c_1 differ only due to the exchange of colors of edges pz_1 and pz_2. Thus it has contain one of the edges pz_1 or pz_2. Therefore it has to be either a $(\eta, 1)$ or $(\eta, 2)$ bichromatic cycle since $d_1(p, z_1) = 2$, $d_1(p, z_2) = 1$. This also means that the bichromatic cycle has to contain the vertex q, since $d(p, q) = \eta$. Thus the bichromatic cycle has to be a $(\eta, 1)$ bichromatic cycle since $2 \notin F_q(d_1)$. This means that $d_1(y_k, y'_k) = 1$. Now recolor the edge xy_k with color α to get a coloring d'_1 of G_1. If d'_1 is valid we are done and $c'_1 = d'_1$. If it is not valid then there has to be a $(\alpha, 1)$ bichromatic cycle containing the vertex x, implying that there existed a $(1, \alpha, xy_k)$ critical path with respect to the coloring d_1 and hence with respect to the coloring $c'_{1,\eta}$. But in view of (5), there already exists a $(1, \alpha, xy_1)$ critical path and by $Fact$ 1, $(1, \alpha, xy_k)$ critical path is not possible, a contradiction. Thus the coloring d'_1 is valid and let $c'_1 = d'_1$.

Thus there exists a valid coloring c'_1 of G_1 such that the coloring $c'_{1,\eta}$ of $G_{1,\eta}$ is derivable from c'_1. $\square$

Now in view of *Claim* 6 and *Claim* 7 there does not exists any $(1, \eta, xy_1)$ critical path with respect to the coloring $c'_{1,\eta}$ of $G_{1,\eta}$, a contradiction to *Critical Path Property* (i.e., *Lemma* 5 or *Lemma* 7).

We conclude that $c_{1,\eta}(p, p') \in \{1, 2\}$.

$\blacksquare$

Assumption 7. *In view of Lemma 17, Lemma 19 and Lemma 20, let z_1 be the vertex such that $\{c_{1,\eta}(p, z_1)\} = \{1, 2\} - \{c_{1,\eta}(p, p')\}$. It follows that $\{c_{j,\eta}(p, z_1)\} = \{1, 2\} - \{c_{j,\eta}(p, p')\}$ and $\{e_1, e_2\} = \{pp', pz_1\}$.*

Observation 4. $\hspace{1em}$ (a) *If $c_{1,\eta}(p, p') = c_{j,\eta}(p, p') = 2$, we have by Assumption 7 that $c_{1,\eta}(p, z_1) = c_{j,\eta}(p, z_1) = 1$. Thus with respect to the partial coloring $c_{1,\eta}$, there exists a $(1, \eta, xy_1)$ critical path of length at least five which contains the vertex z_1. It follows that $c_{1,\eta}(z_1, z'_1) = \eta$ since $z_1 z'_1$ is just the fourth edge of this $(1, \eta, xy_1)$ critical path.*

$\hspace{2em}$ (b) *If $c_{1,\eta}(p, p') = c_{j,\eta}(p, p') = 1$, we have by Assumption 7 that $c_{1,\eta}(p, z_1) = c_{j,\eta}(p, z_1) = 2$. Thus with respect to the partial coloring $c_{j,\eta}$, there exists a $(2, \eta, xy_j)$ critical path of length at least five which contains the vertex z_1. It follows that $c_{j,\eta}(z_1, z'_1) = \eta$ since $z_1 z'_1$ is just the fourth edge of this $(2, \eta, xy_j)$ critical path.*

Local Recolorings: If a partial coloring h of G is obtained from a partial coloring c of G by recoloring only certain edges incident on the vertices belonging to $N_G(p) - \{p', q\} = \{z_1, z_2, \ldots, z_k\}$ and also possibly the edge pp', then h is said to be obtained from c by local recolorings.

The concept of local recolorings turns out to be crucial for the rest of the proof. The following lemma provides the main tool in this respect.

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Lemma 21.   *(a) Let $c_{1,\eta}(p, p') = c_{j,\eta}(p, p') = 2$. Also let $h_{1,\eta}$ be any valid coloring obtained from $c_{1,\eta}$ by recoloring only certain edges incident on the vertices belonging to $N_G(p) - \{p', q\} = \{z_1, z_2, \ldots, z_k\}$ and also possibly the edge pp' (i.e., by only local recolorings). Then there exists a valid coloring h_1 of G_1 such that the valid coloring $h_{1,\eta}$ of $G_{1,\eta}$ is derivable from h_1.*

 (b) Let $c_{1,\eta}(p, p') = c_{j,\eta}(p, p') = 1$. Also let $f_{j,\eta}$ be any valid coloring obtained from $c_{j,\eta}$ by recoloring only certain edges incident on the vertices belonging to $N_G(p) - \{p', q\} = \{z_1, z_2, \ldots, z_k\}$ and also possibly the edge pp' (i.e., by only local recolorings). Then there exists a valid coloring f_j of G_j such that the valid coloring $f_{j,\eta}$ of $G_{j,\eta}$ is derivable from f_j.

Proof.   (a) Recall that $\eta \neq 1, 2$. If $\eta \notin F_x(c_1)$, then $c_{1,\eta} = c_1$. In this case we take $h_1 = c_1$. Otherwise if $\eta \in F'_x(c_1)$, let xy_k be the edge in G_1 such that $c_1(x, y_k) = \eta$. Note that $k \neq 1$. It is enough to show that we can extend the valid coloring $h_{1,\eta}$ of $G_{1,\eta}$ to a valid coloring h_1 of G_1 by assigning an appropriate color to the edge xy_k (Reader may note that neither pp' nor any edge incident on the vertices in $\{z_1, z_2, \ldots, z_k\}$ can be the edge xy_k since $x \notin \{p, p', z_1, z_2, \ldots, z_k, z'_1, z'_2, \ldots, z'_k\}$ due to $Lemma$ 15). Now assign color η to the edge xy_k to get a coloring d. If the coloring d is valid we are done and we have $h_1 = d$. If it is not valid then there has to be a bichromatic cycle created in G_1 with respect to the coloring d. The cycle has to be a (η, θ) bichromatic cycle, where $d(y_k, y'_k) = \theta$. Moreover we can infer that $\theta \in F_x(d)$. If $\theta \neq d(p, p') = 2$, then let $d' = d$. Otherwise we have $\theta = d(p, p') = 2$. Now there exists a color $\omega \neq 2, \eta$ that is a candidate for the edge $y_k y'_k$. Recolor the edge $y_k y'_k$ using color ω to get a coloring d' of G_1. Now if d' is a valid coloring, then we are done and we have $h_1 = d'$. If it is not valid, then $d'(y_k, y'_k) \neq 2$. Let $d'(y_k, y'_k) = \beta \neq 2$. Moreover with respect to the coloring d' there should be a (η, β) bichromatic cycle. Also let $\alpha\ (\neq \eta)\ \in C - F_x(c_1) = C - F_x(d')$. Now if,

(1) $\beta = 1$.

Claim 8. *None of the $(1, \gamma, xy_1)$ critical paths, where $\gamma\ (\neq \eta)\ \in C - F_x(c_1)$ are altered in the process of getting the coloring $h_{1,\eta}$ from $c_{1,\eta}$.*

Proof. Recall that only the edges incident on vertices z_i, where $z_i \in N_G(p) - \{p', q\}$ and edge pp' are possibly recolored to get the coloring $h_{1,\eta}$ of $G_{1,\eta}$ from $c_{1,\eta}$. Note that by these recolorings the $(1, \gamma, xy_1)$ critical path cannot be extended since $x, y_1 \notin \{p, p', z_1, z_2, \ldots, z_k, z'_1, z'_2, \ldots, z'_k\}$ due to $Lemma$ 15. Now if any $(1, \gamma, xy_1)$ critical paths are altered then they have to contain the above mentioned edges. Note that none of the vertices in $\{z_1, z_2, \ldots, z_k\}$ or vertex p' can be the end vertices x or y_1 and hence any critical path containing the vertex z_i or p' should also contain the vertex p since $degree_{G_{1,\eta}}(z_i) = 2$. We can infer that with respect to the coloring $c_{1,\eta}$, the $(1, \gamma, xy_1)$ critical path passes through the vertex p. It follows that this critical path has to contain the edge pz_1 since $c_{1,\eta}(p, z_1) = 1$ (from part (a) of $Observation$ 4). Now since $z_1 \in W_0$ (i.e., $degree_{G_{1,\eta}}(z_1) = 2$), this implies that $c_{1,\eta}(z_1, z'_1) = \gamma$, a contradiciton since from part (a) of $Observation$ 4, we know that $c_{1,\eta}(z_1, z'_1) = \eta$. Thus there cannot be any $(1, \gamma, xy_1)$ critical path containing the edges incident on vertices z_i, where $z_i \in N_G(p) - \{p', q\}$ and edge pp'. Thus none of the $(1, \gamma, xy_1)$ critical paths, where $\gamma \in C - F_x(c_1), \gamma \neq \eta$ are altered. □

Since d' is not valid there has to be a $(\eta, 1)$ bichromatic cycle that passes through the vertex x. Now recolor the edge xy_k with color α to get a coloring d''. Now if still there is a bichromatic cycle, then it should contain the edge xy_k and hence the edge $y_k y'_k$. Therefore it is a $(\alpha, 1)$ bichromtic cycle. This implies by $Fact$ 2 that there existed a $(1, \alpha, xy_k)$ critical path with respect to the coloring d' and hence with respect to the coloring $h_{1,\eta}$. But in view of $Claim$ 8, there exists a $(1, \alpha, xy_1)$ critical path with respect to the coloring $h_{1,\eta}$, a contradiciton in view of $Fact$ 1. Thus the coloring d'' is valid.

(2) $\beta \neq 1$. This implies that $\beta\ (\neq \eta)\ \in F'_x(d')$. Let $y_t \in N'_G(x)$ be such that $d'(x, y_t) = \beta$. Thus $d'(y_t, y'_t) = \eta$. Now recolor the edge xy_k with color $\alpha \in C - F_x(d')$ to get a coloring d''. Note that $\alpha \neq \eta$ since $\eta \notin C - F_x(d')$. Now if still there is a bichromatic cycle, then it should contain the edge xy_k and hence the edge $y_k y'_k$. Therefore it is a (α, β) bichromtic cycle. Thus the bichromatic cycle should contain the edge xy_t. Since $degree_{G_1}(y_t) = 2$, the bichromatic cycle should contain the edge $y_t y'_t$. But by our assumption, $d''(y_t, y'_t) = d'(y_t, y'_t) = \eta \neq \alpha$, a contradiction. Thus the coloring d'' is valid.

Now let $h_1 = d''$. Thus we get a valid coloring of G_1 from $h_{1,\eta}$.

(b) The proof of this is similar to that of part (a) with G_j, c_j, y_j taking the roles of G_1, c_1, y_1 respectively and the colors 1 and 2 exchanging their roles.

∎

Lemma 22.   (a) If $c_{1,\eta}(p, p') = c_{j,\eta}(p, p') = 2$, then with respect to the coloring $c_{1,\eta}$, $2 \notin S_{z_1 z_1'}$. (Recall that by Assumption 7, $\{c_{1,\eta}(p, z_1)\} = \{c_{j,\eta}(p, z_1)\} = \{1, 2\} - \{c_{1,\eta}(p, p')\} = \{1\}$.)

(b) If $c_{j,\eta}(p, p') = c_{1,\eta}(p, p') = 1$, then with respect to the coloring $c_{j,\eta}$, $1 \notin S_{z_1 z_1'}$. (Recall that by Assumption 7, $\{c_{j,\eta}(p, z_1)\} = \{c_{1,\eta}(p, z_1)\} = \{1, 2\} - \{c_{j,\eta}(p, p')\} = \{2\}$.)

Proof.   (a) Suppose not. That is $2 \in S_{z_1 z_1'}$. Note that by part (a) of *Observation* 4, we have $c_{j,\eta}(p, z_1) = 1$ and $c_{j,\eta}(z_1, z_1') = \eta$. Therefore there exists some $\theta \notin \{1, 2, \eta\}$ missing in $S_{z_1 z_1'}$. Now recolor edge $z_1 z_1'$ with color θ to get a coloring $c_{1,\eta}'$. If the coloring $c_{1,\eta}'$ is valid, then let $c_{1,\eta}'' = c_{1,\eta}'$. Otherwise a bichromatic cycle gets formed by the recoloring. Since $c_{1,\eta}'(p, z_1) = 1$, it has to be a $(1, \theta)$ bichromatic cycle and it passes through the vertex p. Thus there exists $z_i \in N_G(p) - \{q, p'\}$ such that $c_{1,\eta}'(p, z_i) = \theta$ and $c_{1,\eta}'(z_i, z_i') = 1$.

Now there exists a color $\mu \notin \{1, \theta, 2, \eta\}$ missing at p. Recolor the edge $p z_1$ with color μ to get a coloring $c_{1,\eta}''$. This clearly breaks the $(1, \theta)$ bichromatic cycle that existed with respect to $c_{1,\eta}'$. But if a new bichromatic cycle gets formed with respect to $c_{1,\eta}''$, then it has to contain vertex z_1 and therefore the edge $z_1 z_1'$, implying that it has to be a (μ, θ) bichromatic cycle since $c_{1,\eta}''(z_1, z_1') = \theta$. This cycle passes through the vertex p and hence passes through the vertex z_i since $c_{1,\eta}''(p, z_i) = \theta$, implying that $c_{1,\eta}''(z_i, z_i') = \mu$, a contradiction since $c_{1,\eta}''(z_i, z_i') = 1$. Thus the coloring $c_{1,\eta}''$ is valid.

Note that we have possibly changed the colors of the edges $p z_1$ and $z_1 z_1'$ to get $c_{1,\eta}''$ from $c_{1,\eta}$ (i.e., only local recolorings are done). Therefore by part (a) of *Lemma* 21 we infer that there exists a coloring c_1'' of G_1 such that $c_{1,\eta}''$ is derivable from c_1''. *It follows from Critical Path Property (i.e., Lemma 5 or Lemma 7) that there exists a $(1, \eta, xy_1)$ critical path with respect to the coloring $c_{1,\eta}''$.* On the other hand recall that with respect to $c_{1,\eta}$ there existed a $(1, \eta, xy_1)$ critical path passing through $p z_1$ and $z_1 z_1'$ (by part (a) of *Observation* 4). But while getting $c_{1,\eta}''$ from $c_{1,\eta}$ we have indeed changed the color of at least one of the edges $p z_1$ or $z_1 z_1'$ using a color other than 1 and η. It follows that the $(1, \eta)$ maximal bichromatic path which contains the vertex x ends at either vertex p or z_1. Noting that $p, z_1 \neq y_1$, we infer by *Fact* 1 that there cannot be a $(1, \eta, xy_1)$ critical path with respect to the coloring $c_{1,\eta}''$, a contradiciton.

(b) The proof of this is similar to that of part (a) with $G_{j,\eta}$, $c_{j,\eta}$, y_j taking the roles of $G_{1,\eta}$, $c_{1,\eta}$ and y_1 respectively and the colors 1 and 2 exchanging their roles.

∎

3.5   Getting a valid coloring that contradicts the Critical Path Property either from c_1 or from c_j

In this section we will get the final contradiction in the following way: If $c_{1,\eta}(p, p') = c_{j,\eta}(p, p') = 1$, then we will show that we can get a coloring c_j' from c_j that contradicts the Critical Path Property. Otherwise if $c_{1,\eta}(p, p') = c_{j,\eta}(p, p') = 2$, then we will show that we can get a coloring c_1' from c_1 that contradicts the *Critical Path Property*.

The two colorings c_1 and c_j are very similar and hence we will only describe the way we get c_1' from c_1. The same agruments can be imitated easily for c_j by keeping the following correspondences in mind.

1. Vertex y_1 has same role as vertex y_j.

2. Colors 1 and 2 exchange their roles.

3. $(1, \gamma, xy_1)$ critical path has the same role as $(2, \gamma, xy_j)$ critical path, for $\gamma \in C'$. The *Critical Path Property* of c_1 corresponds to that of c_j (See Remarks after *Assumption* 3 and *Assumption* 6).

4. Part (a) of *Lemma* 21 and *Lemma* 22 applies to coloring c_1 while part (b) applies to coloring c_j in a corresponding way.

5. *Lemma* 14 has the same role as *Lemma* 13.

We make the following assumption:

Assumption 8. *Let $c_{1,\eta}(p, p') = c_{j,\eta}(p, p') = 2$.*

Observation 5. *In view of Assumption 8, from Observation 4 there exists a $(1, \eta, xy_1)$ critical path which contains the vertex z_1 with respect to the partial coloring $c_{1,\eta}$. Moreover this path is of length at least five. It follows that $c_{1,\eta}(p, z_1) = 1$ and $c_{1,\eta}(z_1, z_1') = \eta$. The first five vertices of the path are x, q, p, z_1, z_1'. Then clearly $z_1' \neq y_1$ and hence is not a pendant vertex in $G_{1,\eta}$. Thus we have $S_{z_1 z_1'} \neq \emptyset$ and $1 \in S_{z_1 z_1'}$.*

Getting a valid coloring d_1 of $G_{1,\eta} - \{pz_1\}$ from $c_{1,\eta}$ by only local recolorings

In view of $Lemma$ 22 and since $c_{1,\eta}(p, z_1) = 1$, the color 2 is a candidate for the edge $z_1 z_1'$. We get a valid coloring d_1 of $G_{1,\eta} - \{pz_1\}$ from $c_{1,\eta}$ by removing the edge pz_1 and recoloring the edge $z_1 z_1'$ by the color 2. Note that d_1 is valid since z_1 is a pendant vertex in $G_{1,\eta} - \{pz_1\}$. Moreover we have broken the $(1, \eta, xy_1)$ critical path. Hence we have,

$$\text{With respect to the partial coloring } d_1, \text{ there does not exists any } (1, \eta, xy_1) \text{ critical path.} \tag{6}$$

Lemma 23. *With respect to the partial coloring d_1 of $G_{1,\eta}$, $\forall \gamma \in C - F_p(d_1)$, there exists a $(2, \gamma, pz_1)$ critical path. Since each of these critical paths has to contain the edge pp', we can infer that $C - F_p(d_1) \subseteq S_{pp'}$.*

Proof. Suppose not. Then there exists a color $\gamma \in C - F_p(d_1)$ such that there is no $(2, \gamma, pz_1)$ critical path. By $Fact$ 2 color γ is valid for the edge pz_1. Thus we get a valid coloring d_1' of $G_{1,\eta}$ by coloring the edge pz_1 with color γ.

Note that we have possibly changed the colors of the edges pz_1 and $z_1 z_1'$ to get d_1' from $c_{1,\eta}$ (i.e., only local recolorings are done). Therefore by part (a) of $Lemma$ 21 we infer that there exists a valid coloring of G_1 from which d_1' can be derived. *It follows from $Critical\ Path\ Property$ (i.e., Lemma 5 or Lemma 7) that there exists a $(1, \eta, xy_1)$ critical path with respect to the coloring d_1'.* On the other hand recall that with respect to $c_{1,\eta}$ there existed a $(1, \eta, xy_1)$ critical path passing through pz_1 and $z_1 z_1'$ (by $Observation$ 5). But while getting d_1' from $c_{1,\eta}$ we have indeed changed the color of the edges $z_1 z_1'$ using the color $2 \notin \{1, \eta\}$. It follows that the $(1, \eta)$ maximal bichromatic path which contains the vertex x ends at either vertex p or z_1. Noting that $p, z_1 \notin y_1$, we infer that there cannot be a $(1, \eta, xy_1)$ critical path with respect to the coloring d_1', a contradiciton.
∎

Note that with respect to $G_{1,\eta} - \{pz_1\}$, $|F_p(d_1)| \leq \Delta - 1$ and therefore $|C - F_p(d_1)| \geq 2$. But we know that color $1 \notin F_p(d_1)$. Since $|C - F_p(d_1)| \geq 2$, there exists a color $\mu \neq 1 \in C - F_p(d_1)$. Note that $\mu \neq 2$ also. The following observation is obvious in view of $Claim$ 23:

Observation 6. *With respect to the partial coloring d_1 of $G_{1,\eta}$, $1, \mu \notin F_p(d_1)$ and there exist $(2, 1, pz_1)$ and $(2, \mu, pz_1)$ critical paths.*

Selection of a special color θ: Since $|F_{p'}(d_1)| \leq \Delta$, there exists a color θ missing at vertex p'. By $Lemma$ 23, $\theta \notin C - F_p(d_1) \subseteq F_{p'}(d_1)$. Thus $\theta \in F_p(d_1)$. Clearly $\theta \neq 2$ since $2 \in F_{p'}$ and $\theta \neq 1, \mu$ because $1, \mu \notin F_p(d_1)$ and hence by $Lemma$ 23 we have $1, \mu \in S_{pp'}(d_1)$. Further $\theta \neq \eta$. This is because by $Lemma$ 14, the $(\eta, 2)$ maximal bichromatic path starts at vertex q and contains the vertex y_j'. Clearly the first three vertices of this path are q, p, p'. Recall that the length of the segment of this path between vertices q and y_j' is at least three. Therefore $\eta \in S_{pp'}(d_1)$. Now without loss of generality let $d_1(p, z_2) = \theta (\neq 1, \eta, \mu, 2)$. Note that z_2 is a vertex different from z_1.

Note that with respect to the coloring $c_{1,\eta}$, the $(1, \eta, xy_1)$ critical path passes through the vertex z_1 (by (5)). This critical path cannot contain the vertex z_2. This is because if z_2 is an internal vertex of this critical path, then the edge pz_2 should be contained in the path, a contradiction since $c_{1,\eta}(p, z_2) = \theta \neq 1, \eta$. On the other hand if z_2 is an end vertex then it implies that $z_2 \in \{x, y_1\}$, a contradiction in view of $Lemma$ 15. Thus vertex z_2 is not contained in the $(1, \eta, xy_1)$ critical path. While getting the coloring d_1 from $c_{1,\eta}$, this path was broken due to the recoloring of $z_1 z_1'$ and pz_1. It follows that the $(1, \eta)$ maximal bichromatic path that starts at vertex y_1 does not contain vertex z_2. Thus we can infer that,

Observation 7. *With respect to the coloring d_1, there cannot exist a $(1, \eta, y_1, z_2)$ maximal bichromatic path.*

Getting a valid coloring d_2 of $G_{1,\eta} - \{pz_2\}$ from d_1 of $G_{1,\eta} - \{pz_1\}$ by only local recolorings

We get a coloring d_1'' of $G_{1,\eta} - \{pz_1, pz_2\}$ from d_1 by discarding the edge pz_2. Note that the partial coloring d_1'' of $G_{1,\eta}$ is valid.

Now recolor the edge pz_1 with color the *special color* θ to get a coloring d_2 of $G_{1,\eta} - \{pz_2\}$. Note that the color θ is a candidate for the edge pz_1 with respect to the coloring d_1'' since $d_1''(z_1, z_1') = 2$ and $\theta \notin F_p(d_1'')$ since we have removed the edge pz_2 (Recall that $d_1''(p, z_2) = \theta$). We claim that d_2 is valid also. Clearly if there is any bichromatic cycle created, then it has to be a $(\theta, 2)$ bichromatic cycle since $d_2(z_1, z_1') = 2$. Now this bichromatic cycle has to pass through vertex p' since $d_2(p, p') = 2$. But by the definition of color θ, it was not present at vertex p'. Thus there cannot be a $(\theta, 2)$ bichromatic cycle. It follows that the partial coloring d_2 of $G_{1,\eta} - \{pz_2\}$ is valid. Recall that by (6) that there exists no $(1, \eta, xy_1)$ critical path with respect to d_1. Note that to get d_2 from d_1, we just assigned $\theta(\neq 1, 2, \eta, \mu)$ to the edge pz_1 and removed the edge pz_2. Thus there is no chance of $(1, \eta, xy_1)$ critical path getting created with respect to d_2. Hence we have,

> *With respect to the partial coloring d_2, there does not exists any $(1, \eta, xy_1)$ critical path.* $\hspace{1em}$ (7)

Getting a valid coloring $c_{1,\eta}'$ of $G_{1,\eta}$ from d_2 of $G_{1,\eta} - \{pz_2\}$ by only local recolorings

Now we will show that we can give a valid color for the edge pz_2 to get a valid coloring for the graph $G_{1,\eta}$. We claim the following:

Lemma 24. *With respect to the coloring d_2 at least one of the colors 1, μ is valid for the edge pz_2. (Recall that by Observation 6, $1, \mu \notin F_p(d_1)$ and therefore $1, \mu \notin F_p(d_2)$)*

Proof. Let $d_2(z_2, z_2') = \sigma$. Now if,

1. $\sigma = 2$. Recolor the edge pz_2 using color 1 to get a coloring d_3. The coloring d_3 is valid because if a bichromatic cycle gets formed it has to be $(1, 2)$ bichromatic cycle containing the vertex p implying that there was a $(2, 1, pz_2)$ critical path with respect to d_2. But by Observation 6, there was a $(2, 1, pz_1)$ critical path with respect to the coloring d_1 and hence with respect to the coloring d_2 (Note that to get d_2 from d_1, we just assigned $d_1(p, z_2) = \theta$ $(\neq 1, 2, \eta, \mu)$ to edge pz_1 and removed the edge pz_2. Thus the $(2, 1, pz_1)$ critical path is not altered during this recoloring). Thus in view of Fact 1, there cannot be any $(2, 1, pz_2)$ critical path with respect to d_2 since $z_1 \neq z_2$, a contradiction. Thus the coloring d_3 is valid.

2. $\sigma \in \{1, \mu\}$. Recolor the edge pz_2 using color $\{1, \mu\} - \{\sigma\}$ to get a coloring d_3. The coloring d_3 will be valid because if a bichromatic cycle gets formed it has to be $(1, \mu)$ bichromatic cycle containing the vertex p. But since color $\sigma \in \{1, \mu\}$ is not present at vertex p, such a bichromatic cycle is not possible.

3. $\sigma \notin \{1, 2, \mu\}$. Recolor the edge pz_2 using color 1 to get a coloring d_2'. If the coloring d_2' is valid, then let $d_3 = d_2'$. Otherwise if the coloring d_2' is not valid, then there has to be a $(\sigma, 1)$ bichromatic cycle. Now let $d_2'(p, z_j) = \sigma$. Then the bichromatic cycle passes through the vertex z_j and hence $d_2'(z_j, z_j') = 1$, since $degree_G(z_j) = 2$. Now we recolor edge pz_2 with color μ to get a coloring d_3. If there is a bichromatic cycle formed with respect to the coloring d_3, then it has to be a (μ, σ) bichromatic cycle and hence it passes through the vertex z_j. But color μ is not present at z_j since $d_2'(z_j, z_j') = 1$. Thus there cannot be any (μ, σ) bichromatic cycle. Hence the coloring d_3 is valid.

Thus either color 1 or μ is valid for the edge pz_2.

$\hspace{1em}$ ∎

To get the coloring d_3 from d_2 we have only given a valid color for the edge pz_2 and have not altered the color of any other edge (i.e., only local recolorings are done). Recall that d_2 does not have any $(1, \eta, xy_1)$ critical path (by (7)). Note that $d_3(x, q) = 1$ and $d_3(q, p) = \eta$. If we give color $\mu \neq 1, \eta$ to the edge pz_2, there is no chance of a $(1, \eta, xy_1)$ critical path getting formed in d_3. On the other hand, by giving color 1 to the edge pz_2 if a $(1, \eta, xy_1)$ critical path gets formed, then it means that there exists a $(1, \eta, y_1, z_2)$ maximal bichromatic path with respect to d_2 and hence with respect to d_1. But by *Observation* 7 such a bichromatic path does not exist. Now let $c_{1,\eta}' = d_3$. Thus we have,

> *With respect to the valid coloring $c_{1,\eta}'$ of $G_{1,\eta}$, there does not exists any $(1, \eta, xy_1)$ critical path.* $\hspace{1em}$ (8)

In getting $c'_{1,\eta}$ from $c_{1,\eta}$ we have done only local recolorings and thus by $Lemma$ 21 $c'_{1,\eta}$ can be derived from some valid coloring c'_1 of G_1. Note that we have not changed the color of the edge $y_1 y'_1$ while getting $c'_{1,\eta}$ from $c_{1,\eta}$ since $y_1 \notin \{p, p', z_1, \ldots, z_k, z'_1, \ldots, z'_k\}$ (by $Lemma$ 15). Thus $c'_{1,\eta}(y_1, y'_1) = 1$. It follows that the $Critical\ Path\ Property$ of $c'_{1,\eta}$ is the same as $Critical\ Path\ Property$ of $c_{1,\eta}$. This implies that there exists a $(1, \eta, xy_1)$ critical path with respect to the coloring $c'_{1,\eta}$, a contradiction in view of (8).

This completes the proof. ∎

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